ALGORITHM FOR TRANSFORMATION OF WIND ROSE BY INCREASING OR DECREASING OF COMPASS DIRECTIONS

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Abstract. The necessity to transform the standard (for Bulgaria) climatic 8-point wind roses, named *Observed Wind Roses* (OWR), was described in previous paper of the authors - Gromkova at al. The algorithm of calculation of the so-called *Primitive Wind Rose* (PWR) and further on the *Transformed Wind Rose* (TWR) in different number of compass directions then 8 is the purpose of the present paper.

Keywords: air pollution, wind rose.

Abbreviations:

OWR – *Observed Wind Roses*
TWR – *Transformed Wind Rose*
DWR – *Discrete Wind Rose*
PWR – *Primitive Wind Rose*
c.d.f. – *cumulative distribution function*
p.d.f. – *probability density function*

Introduction

The standard wind roses are represented with 8-compass directions – one for each 45º of the horizon, in the Climatic reference book of Bulgaria (1982). It was shown in a previous paper (Gromkova et al. 20 ..) that using a standard wind rose the determination of the annual mean air pollution concentration field from a point source is not adequate. For this reason the presented algorithm to transform the 8-point wind rose to a rose with more (or less, if it necessary) compass direction was created.

In Gromkova at al., 20… a wind direction was adopted as stochastic event and a random variable X with a corresponding *cumulative distribution function* – $F(x)$ and a *probability density function* – $p(x,\theta)$ was associated to it. In the present paper an algorithm for obtaining the *probability density function* – $p(x,\theta)$ (p.d.f.) of the so-called *Observed Wind Roses* (OWR) – the rose from the Climatic reference book of Bulgaria (1982), and then to calculate the so-called *Transformed Wind Rose* in different compass directions (usually more) is presented.

Determination of the parameter θ of the function $p(x,\theta)$

Variant 1

The function $p(x,\theta)$ – *probability density function* – in this variant has the form:

$$p(x,a_i,b_i) = a_i + b_i x \qquad (1)$$

in the intervals $(x_i, x_{i+1}]$. Between every two compass directions there is only one straight line. There are $2n$ unknowns: a_i – the y-intercept, and b_i – the slope of the line, if the compass directions are n. A method for determining the unknowns is given below:

Replacing $N(x_i)$ and $N(x_1)$ in the integrals of formulas (1) and (2) in Gromkova at al, 20…., by the empirical estimates $\hat{N}_i$ and $\hat{N}_1$, and substituting x_i by $x_i = 2(i-1)t$ in the limits of the integrals, the following expressions are obtained:

$$\hat{N}_i = \int_{(2i-3)t}^{2(i-1)t} (a_{i-1} + b_{i-1} z)\, dz + \int_{2(i-1)t}^{(2i-1)t} (a_i + b_i z)\, dz, \quad i = 2, 3, \dots, n,$$

$$\hat{N}_1 = \int_{(2n-1)t}^{2nt} (a_n + b_n z)\, dz + \int_{0}^{t} (a_1 + b_1 z)\, dz \quad i = 1\ (x_1 = 0).$$

After integration and some transformations the following relations are reached:

$$M_i = d_{i-1} + d_i + (4i-5)b_{i-1} + (4i-3)b_i, \quad i = 2, 3, \dots, n, \qquad (2)$$

$$M_1 = d_n + d_1 + (4n-1)b_n + b_1, \quad i = 1, \qquad (3)$$

where $M_i = 2\hat{N}_i / t^2$ and $d_i = 2a_i / t$.

The condition the strength lines defined with (1) to have equal values on both sides of the compass direction points (Gromkova et al, 20….) is:

$$d_i + 4(i-1)b_i = d_{i-1} + 4(i-1)b_{i-1}, \quad i = 2, 3, \dots, n, \qquad (4)$$

$$d_1 = 4nb_n + d_n, \quad i = 1. \qquad (5)$$

The equations (2), (3), (4) and (5) represent a system of $2n$ linear equations in $2n$ unknowns. The solution of the system in brief follows:

The differences $M_{i+1} - M_i$ and the sums $d_{i+1} + d_i$ are made:

$$M_{i+1} - M_i = d_{i+1} - d_{i-1} - (4i-5)b_{i-1} + 2b_i + (4i+1)b_{i+1}, \quad i = 2, 3, \dots, n-1, \qquad (6)$$

$$M_2 - M_1 = d_2 - d_n - (4n-1)b_n + 2b_1 + 5b_2, \quad i = 1, \qquad (7)$$

$$d_{i+1} - d_{i-1} = 4(i-1)b_{i-1} + 4b_i - 4ib_{i+1}, \quad i = 2, 3, \dots, n-1, \qquad (8)$$

$$d_2 - d_n = 4b_n + 4b_1 - 4b_{i+1}, \quad i = 1. \qquad (9)$$

The differences from (6) and (7) are replaced in the differences (8) and (9). A system of (n - 1) equations in n unknowns in respect of b_i is obtained:

$$M_{i+1} - M_i = b_{i-1} + 6b_i + b_{i+1}, \quad i = 2, 3, \ldots, n-1, \tag{10}$$

$$M_2 - M_1 = b_n + 6b_1 + b_2, \quad i = 1. \tag{11}$$

Summarizing the equations (4) and (5) for i from 1 to n, the following expression is reached:

$$\sum_{i=1}^{n} b_i = 0 .$$

Adding the last equality to (10) and (11) a system of n equations in n unknowns in respect of b_i is obtained.

The system is presented in a matrix form as $\boldsymbol{A.B} = \boldsymbol{F}$, where $\boldsymbol{F}$ is a column-vector with components $f_i = M_{i+1} - M_i$, for $i = 1, 2, \ldots, n-1$, and $f_n = 0$, $\boldsymbol{B}$ is a column-vector with components the unknowns b_i, and $\boldsymbol{A}$ is a quadratic matrix of order n of the system coefficients. The system can be written as follows too:

$$\begin{pmatrix} u & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & u & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & u & 1 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & u & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & u & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \cdots \\ b_{n-2} \\ b_{n-1} \\ b_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \cdots \\ f_{n-2} \\ f_{n-1} \\ 0 \end{pmatrix}.$$

The diagonal elements, marked with u, are equal to 6. For obtaining the inverse matrix $\boldsymbol{A}^{-1}$ the matrix $\boldsymbol{A}$ is divided in 4 blocks:

$$\boldsymbol{A} = \begin{pmatrix} \alpha_{11} \; \alpha_{12} \\ \alpha_{21} \; \alpha_{22} \end{pmatrix},$$

where α_{11} is a quasidiagonal matrix (Jacobian) of order $n-1$ with main diagonal elements equal to 6 and with adjacent off-diagonal elements equal to 1, α_{12} is a column-vector with components $(\alpha_{12})_1 = 1$ and $(\alpha_{12})_{n-1} = 1$, and remaining elements equal to 0, α_{21} is a row-vector with components equal to 1, and α_{22} is a scalar equal to 1, too.

The inverse matrix is represented in blocks, too (Demidovitch and Maron, 1960):

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \beta_{11} \; \beta_{12} \\ \beta_{21} \; \beta_{22} \end{pmatrix}$$

where:

$$\beta_{11} = \alpha_{11}^{-1} - \beta_{12}\alpha_{21}\alpha_{11}^{-1}, \quad (12)$$

$$\beta_{12} = \alpha_{11}^{-1}\alpha_{12}(\alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12})^{-1}, \quad (13)$$

$$\beta_{21} = -\beta_{22}\alpha_{21}\alpha_{11}^{-1}, \quad (14)$$

$$\beta_{22} = (\alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12})^{-1}. \quad (15)$$

In the expressions above α_{11}^{-1} is the inverse matrix of α_{11}. As the matrix α_{11} is symmetrical, the inverse matrix α_{11}^{-1} is symmetrical, too. Let's denote the elements of α_{11}^{-1} by c_{ij} (i, j = 1, 2, … , n – 1). It is possible to prove (see Appendix), that the analytical expression of c_{ij} is represented by the formula:

$$c_{ij} = -\frac{(Q^i - P^i)(Q^{n-j} - P^{n-j})}{(Q-P)(Q^n - P^n)}, \quad i \le j \le n-1, \quad (16)$$

where

$$P = (\sqrt{u^2-4} - u)/2 \quad \text{and} \quad Q = -(\sqrt{u^2-4} + u)/2$$

Using the formula for c_{ij} it can be demonstrated that the elements of the matrices (blocks) – $(\beta_{11})_{ij}$, $(\beta_{12})_i$, $(\beta_{21})_j$ and (β_{22}) have the form:

$$(\beta_{11})_{ij} = \frac{Q(Q^j - 1)(Q^{i-j} - Q^{n-i})}{(Q^2-1)(Q^n-1)}, \quad j < i \le n-1,$$

$$(\beta_{11})_{ij} = \frac{Q(Q^{n-j} - 1)(Q^{j-i} - Q^{i})}{(Q^2-1)(Q^n-1)}, \quad i \le j \le n-1,$$

$$(\beta_{12})_{i} = \frac{(Q-1)(Q^{n-i} + Q^{i})}{(Q+1)(Q^n-1)}, \quad i \le n-1,$$

$$(\beta_{21})_{j} = \frac{Q(Q^{n-j} - 1)(Q^{j} - 1)}{(Q^2-1)(Q^n-1)}, \quad j \le n-1,$$

$$(\beta_{22}) = \frac{(Q-1)(Q^{n} + 1)}{(Q+1)(Q^n-1)}.$$

The elements of the inverse matrix $\boldsymbol{A}^{-1}$ can be determined by using the formulas of the blocks β_{11}, β_{12}, β_{21} and β_{22} – the expressions (12), (13), (14) and (15). Finally, after the multiplication $\boldsymbol{B} = \boldsymbol{F}.\boldsymbol{A}^{-1}$ the strength line coefficients b_i are determined according to the next expression:

$$b_i = \frac{Q}{(Q^2-1)(Q^n-1)} \times \left[\sum_{j=1}^{i-1} (Q^j - 1)(Q^{i-j} - Q^{n-i}) f_j + \sum_{j=i}^{n} (Q^{n-j} - 1)(Q^{j-i} - Q^i) f_j \right], \quad i = 1, \ldots, n.$$

To obtain the unknown d_i (respectively a_i), d_n is defined by the expression (5), then it is substituted in (3) and d_1 is represented by:

$$d_1 = (M_1 + b_n - b_1) / 2.$$

The remaining unknowns d_i for $i = 2, \ldots, n$ are defined recursively by the formula (8):

$$d_i = d_{i-1} + 4(i-1)(b_{i-1} - b_i).$$

The coefficients a_i are obtained by the following expression:

$$a_i = t\, d_i / 2.$$

Variant 2

The analytical expression of the function $p(x,\boldsymbol{a})$ in this case has the form:

$$p(x,a_i) = a_i,$$

defined in the intervals $(x_i - t, x_i + t]$, for $i = 2, 3, \ldots, n$, and in the intervals $(360 - t, 360]$ and $(0,t]$ for $i = 1$, i.e. for $x_1 = 0$. The unknowns are n for n directions.

The determination of these constants is as follows:

$$a_i = \hat{N}_i / 2t .$$

Appendix

Let c_{ij} are the elements of the inverse matrix α_{11}^{-1}. Then the following matrix equality is satisfied:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1,n-3} & c_{1,n-2} & c_{1,n-1} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2,n-3} & c_{2,n-2} & c_{2,n-1} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3,n-3} & c_{3,n-2} & c_{3,n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-3,1} & c_{n-3,2} & c_{n-3,3} & \cdots & c_{n-3,n-3} & c_{n-3,n-2} & c_{n-3,n-1} \\ c_{n-2,1} & c_{n-2,2} & c_{n-2,3} & \cdots & c_{n-2,n-3} & c_{n-2,n-2} & c_{n-2,n-1} \\ c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \cdots & c_{n-1,n-3} & c_{n-1,n-2} & c_{n-1,n-1} \end{pmatrix} \times \begin{pmatrix} u & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & u & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & u & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & u & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & u & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & u \end{pmatrix} = \boldsymbol{E},$$

where $\boldsymbol{E}$ is the unit matrix. After multiplication of the i-th row of α_{11}^{-1} by all columns of α_{11}, the following equations are obtained:

$$uc_{i1} + c_{i2} = 0,$$
$$c_{i1} + uc_{i2} + c_{i3} = 0,$$
$$\dots$$
$$c_{ii\text{-}1} + uc_{ii} + c_{i,i+1} = 1, \qquad (17)$$
$$c_{ii} + uc_{i,i+1} + c_{i,i+2} = 0,$$
$$\dots$$
$$c_{i,n\text{-}3} + uc_{i,n\text{-}2} + c_{i,n\text{-}1} = 0,$$
$$c_{i,n\text{-}3} + uc_{i,n\text{-}2} + c_{i,n\text{-}1} = 0,$$
$$c_{i,n\text{-}2} + uc_{i,n\text{-}1} = 0. \qquad (18)$$

It is obvious that for $i = 2, 3, \dots, n–3$, $j > i$, c_{ij} satisfies the recurrence relation:

$$c_{i,j\text{-}1} + uc_{ij} + c_{i,j+1} = 0. \qquad (19)$$

To obtain the common term of this recursion, its characteristic equation is used (Markushevitch, 1975):

$$q^2 + uq + q = 0.$$

The roots of this equation, denoted by P and Q, are:

$$P = (\sqrt{u^2 - 4} - u)/2 \text{ and } Q = -(\sqrt{u^2 - 4} + u)/2\,.$$

They are real when $u \geq 2$, i.e. in the discussed case, because $u = 6$: $Q = -(\sqrt{8} + 3) \approx$ -5.8284. The common term of the relation (19) is presented by the expression (Markushevitch, 1975):

$$c_{ij} = G_iP^{j–i} + H_iQ^{j–i}, \quad i = 2, 3, \dots, \quad n – 3, j > i. \qquad (20)$$

The coefficients G_i and H_i are different for each i, i.e. for every row.

The first two terms of the above expression – for $j = i$ and $j = i + 1$ – have the form:

$$c_{ii} = G_i + H_i, \qquad (21)$$

$$c_{i,i+1} = G_iP + H_iQ, \qquad (22)$$

and the last two – for $j = n–2$ and $j = n–1$:

$$c_{i,n\text{-}2} = G_iP^{n–2–i} + H_iQ^{n–2–i}, \qquad (23)$$

$$c_{i,n\text{-}1} = G_iP^{n–1–i} + H_iQ^{n–1–i}. \qquad (24)$$

To determine the unknown coefficients G_i and H_i the equalities (17), written in the form:

$$uc_{ii} + c_{i,i+1} = D_{ii\text{-}1}, \qquad (25)$$

and the expression (18), that are not terms of the recurrence relation (19), are used too. $D_{i,i\text{-}1}$ in the above expression according (17) is $D_{i,i\text{-}1} = 1 - c_{i,i\text{-}1}$. As the matrix $\alpha_{11}{}^{-1}$ is symmetric $c_{i,i\text{-}1} = c_{i\text{-}1,i}$. However the element $c_{i\text{-}1,i}$ is determined from the recurrence relation by multiplication of the i–1-th row and therefore $D_{i,i\text{-}1}$ is determined too.

It is known that $u = -(P + Q)$ and $PQ = 1$ for the roots of the quadratic equation. Replacing (21) and (22) in (25), (23) and (24) in (18) and u in (25) and (18) and after some processing the following system of equations for G_i and H_i is obtained:

$$G_iQ + H_iP = -D_{i,i\text{-}1},$$

$$G_iP^{n-i} + H_iQ^{n-i} = 0.$$

The solution of the system in respect to G_i and H_i is:

$$G_i = -\frac{Q^{n-i}}{Q^{n+1-i} - P^{n+1-i}} D_{i,i-1}, \qquad (26)$$

$$H_i = \frac{P^{n-i}}{Q^{n+1-i} - P^{n+1-i}} D_{i,i-1}. \qquad (27)$$

Determining $D_{i,i\text{-}1}$,.for i = 2, 3, … and s.o., it can be shown that

$$D_{i,i-1} = \frac{(Q^i - P^i)(Q^{n+1-i} - P^{n+1-i})}{(Q-P)(Q^n - P^n)}.$$

After replacing $D_{i,i\text{-}1}$ in (26) and (27) and then G_i and H_i in (26), the expression (16) in the main text – the formula for the coefficients c_{ij} – is reached.

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Алгоритъм за преобразуване на розата на вятъра чрез увеличаване или намаляване на румбовете

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Резюме.. Необходимостта да се преобразува стандартната за България климатична 8-румбова роза на вятъра (наречена тук *Наблюдателна Роза на Вятъра* (НРВ)) възникна във връзка с определянето на концентрацията на климатичното (средногодишното) замърсяване на въздуха от точков източник, което е основна цел на оценките на въздействие върху околната среда (т.н. ОВОС). В предната публикация (Gromkova at al.) бяха представени редица теоретични примери за преимуществата на разработения алгоритъм за трансформиране на розата на вятъра при получаване на полето на приземните концентрации, в сравнение с използването на 8-румбовата роза, дадена в климатичния справочник. Тук е представен математическия апарат, използван за получаване на рози с различен брой румбове.