ON THE FLOW IN THE EARTH'S LIQUID CORE

A.Anufriev [1], *M.Tassev*[2]

[1]Geophysical Institute, ul.Acad.G.Bonchev, bl3, Sofia1113, Bulgaria, e-mail: anufriev@abv.bg
[2] South-West University, Department of Mathematics, ul.Iv.Mihailov 66, Blagoevgrad,2700, Bulgaria, e-mail: matassev@abv.bg

Abstract. The momentum equation for the flow in the Earth's liquid core which depends on two small parameters (the Ekman, E, and the Rossby, R_o numbers), is transformed to a form which depends on other two parameters: $\sqrt{E}$ and $\lambda = \sqrt{E}/R_o$. The first of them is still small, but the other one exceeds $\sqrt{E}$ approximately with six orders of magnitude. The Ekman suction boundary conditions for the inviscid flow in the bulk of the Earth's core also depend on the first parameter only. Therefore we search for the solution of the hydrodynamic problem in the form of the decomposition in the small parameter $\sqrt{E}$. The equations obtained show that the flow of the leading approximation in $\sqrt{E}$ decomposition depends on the only parameter λ. This means that for fast rotating fluids $\lambda = \sqrt{2\Omega\nu}/V_1$ becomes the universal parameter (as e.g. the Reynolds number $R = V_1 L_1/\nu$ in the hydrodynamics) and all flows can be classified only by it. The whole flow is presented as a sum of the force and the geostrophic flows. The first one can be directly determined in terms of integrals of the force. The second one obeys the equations which depend on time derivative of the geostrophic velocity. So the geostrophic flow is the only part of the whole flow which has its own time behavior. It evolves together with magnetic field and temperature distributions. In the opposite, the force flow fits instantly to the force and its time behavior is defined entirely by that of the force distribution. There are two (the inviscid $\lambda \to 0$ and the viscous controlled $\lambda \to \infty$) limits in which the whole flow does not depend on λ and hence on E. Therefore the flow converts in Taylor state in these limits. Depending on the viscosity, the parameter λ for the Earth's core changes from 2.3×10^{-2} for the kinematic viscosity to 3.3×10^{1} for the greatest value of the turbulent viscosity $\nu_T = \eta$ where η is the magnetic diffusivity. So both types of the Taylor state flows can be realized in the Earth's core. From computer simulations point of view the viscous controlled case is especially important, because all the simulations use implicitly λ » 1. For example, Glatzmaier and Roberts (1995, 1996) in fact carried out their simulations for $\lambda \sim 10^3$. Thus their asymptotic (at $\lambda \to \infty$) solution is approximately suitable for the Earth's core conditions with the “turbulent” $\lambda = 3.3\times10^{1}$. It is not trivial, since if we adopt the same values of $\nu(=\nu_T=\eta)$ and V_1 for Venus where Ω is with two orders of magnitude smaller, then λ becomes of order of 1 and the viscous controlled numerical results become unapplicable to this planet. Moreover, being independent of λ (and respectively of E), the viscous controlled computer simulated flows convert into Taylor state and so describe adequately the flow in the Earth's core for the turbulent viscosity. This is also a non-trivial conclusion, taking into account that the typical values of the Ekman numbers in computer simulations typically exceed these in the Earth core with ten orders! Though the computer simulations describe satisfactory the flows with large λ, they are unapplicable to solve the problem with smaller values of λ and especially in the inviscid limit $\lambda\to 0$, due to the necessity of resolution of the thin boundary layers. We believe that this difficulty can be overcome by creating computer codes on the base of the equations presented here which are free of this necessity.

Key words: geodynamo, Taylor state, inviscid limit

Introduction

The flow in the liquid core of the Earth is driven by the Archimedean and the Lorentz forces. This area is situated between the inner core boundary (ICB) and the core mantle boundary (CMB). Determination of this flow is an important part of the whole geodynamo problem. The dimensionless momentum and continuity equations for it can be written in a form:

$$R_o\,\bar{\rho}\left[\frac{\partial \mathbf{V}}{\partial t} + \mathbf{W}\times\mathbf{V}\right] = -\bar{\rho}\,\nabla P - \bar{\rho}\,\mathbf{1}_z\times\mathbf{V} + \mathbf{F} + \bar{\rho}\,E\,\Delta\mathbf{V}, \qquad \nabla\cdot\bar{\rho}(r)\mathbf{V} = 0, \qquad (1.1,2)$$

where $\mathbf{W} = \nabla\times\mathbf{V}$, $\mathbf{F} = \mathbf{F}^a + \mathbf{F}^b$, $P = p/\bar{\rho} + \mathbf{V}^2/2$ and p is the pressure. Here $\mathbf{F}^a = r\,A\,(\bar{\rho}(r)\,\bar{g}(r)/r)\,C$ is the Archimedean and $\mathbf{F}^b = R_m \mathbf{J}\times\mathbf{B}$ is the Lorentz forces where $\mathbf{J} = \nabla\times\mathbf{B}$ and C and **B** are the super-adiabatic temperature and the magnetic field strength. (Generally C and be considered as co-density). The dimensionless parameters, the Ekman, the Rossby and the Reynolds magnetic numbers are defined respectively as

$$E = \frac{\nu}{2\,\Omega\,L_1^2}, \quad R_o = \frac{V_1}{2\,\Omega\,L_1}, \quad R_m = \frac{V_1\,L_1}{\eta} \qquad (1.3)$$

where L_1, V_1, L_1/V_1, $\sqrt{2\Omega\eta\bar{\rho}_1\mu_o}$, ρ_1, $2\Omega\bar{\rho}_1 V_1$ and $2\Omega\bar{\rho}_1 V_1 L_1$ are the space, the flow velocity, the time, the magnetic field strength, the density, the force and the pressure scales. Note that the viscous force in compressible liquid, generally speaking, differs from this in (1.1). However this difference is essential only outside the thin layers where the viscous force is negligible.

The Ekman number in the Earth's core is extremely small: $E \sim 10^{-15}$. This leads to great difficulties in the computer resolution of the thin layers ($\delta = O(\sqrt{E})$) on the core's boundaries. That is why E in the existing computer simulations is typically accepted to be several orders of magnitude greater than its real value (see e.g. Glatzmaier and Roberts (1995, 1996a), Jones et al (1995) Fearn and Morrison (2001) who accept E to be in the range (10^{-3} to 10^{-5}).

The value of E above is obtained for the kinematic viscosity. The turbulent viscosity enhances the value of the Ekman number with several orders of magnitude. However, even the turbulent E hardly exceeds 10^{-10}. A question of principle arises here: whether the flow of the computer simulations carried out for the relatively large values of the Ekman number keep valid for its small values? And yet another question. Is it possible to reformulate the numerical flow problem in a way which could be free of the difficulties connected to the resolution of the thin boundary layers? Here we are trying to answer both questions.

In our further analysis we assume that the force in the bulk of the core does not change during fitting of the flow to it. The magnetic field and the temperature distribution evolve together with the flow, changing the magnetic and the Archimedean forces. Why do we think that the process of adjusting of the flow can be separated from the change of the magnetic field and the temperature distribution? The reason is that the flow changes defined by the enormous ($\sim R_o^{-1}$) accelerations in (1.1), are much faster than these for the field and the temperature.

This argumentation fails for the thin layers where the magnetic diffusion time $\tau_d \sim \delta^2/\eta$ is very small. For the Ekman layer, $\tau_d \sim (\nu/2\Omega)/\eta$. In the case of turbulent viscosity this time is of order of few hours, if we adopt $\nu_{turb} \sim \eta$.

So, when solving the momentum equation we can neglect the changes of the field in the bulk of the core, but we must take them into account in the thin layers. Thus we assume that during the flow fitting, the magnetic field remains unchanged only in the interior of the liquid core, but not in the thin boundary and space layers. The magnetic field in the layers fits together with the flow. Respectively, the boundary conditions (2.4) for the flow outside the boundary layer on its outer surface are obtained by solving the momentum and the induction equations into the layer.

Equation and boundary conditions

The momentum equation (1.1) depends on two small parameters: the Ekman and the Rossby numbers. Let us transform this equation dividing it by $\sqrt{E}$:

$$\frac{\bar{\rho}}{\lambda}\frac{\partial \mathbf{V}}{\partial t} = \frac{-\bar{\rho}\,\nabla P - \mathbf{1}_z \bar{\rho} \times \mathbf{V} + \mathbf{F}}{\sqrt{E}} - \frac{\bar{\rho}}{\lambda}\mathbf{W}\times\mathbf{V} + \sqrt{E}\Delta\mathbf{V}\ , \quad \nabla\cdot\bar{\rho}(r)\mathbf{V} = 0 \qquad (2.1,2)$$

Then new momentum equation depends on other two parameters:

$$\sqrt{E} = 3.5\times10^{-5}\sqrt{\frac{\nu}{\eta}} \quad \text{and} \quad \lambda = \frac{\sqrt{E}}{R_o} = \frac{\sqrt{2\Omega\nu}}{V_1} = 3.3\times10^{1}\sqrt{\frac{\nu}{\eta}} \qquad (2.3)$$

In these estimations we have used the following parameters from Roberts and Glatzmaier (2000): the typical value of the westward drift $V_1 = 5\times10^{-4}\,\mathrm{ms}^{-1}$ and the value of the magnetic diffusivity $\eta = 2\mathrm{m}^2\mathrm{s}^{-1}$. As a space scale, the radius of the liquid core of the Earth $L_1 = r_2 = 3.4\times10^6\,\mathrm{m}$, is used.

We assume that the turbulent viscosity is not smaller than $\nu_M = 10^{-6} m^2 s^{-1}$ and does not exceed $\eta = 2\,\mathrm{m}^2\,\mathrm{s}^{-1}$.

Then λ is situated in the interval: $2.3\times10^{-2} \le \lambda \le 3.3\times10^{1}$. For all possible values of the turbulent viscosity λ exceeds $\sqrt{E}$ with approximately six orders. Therefore, we will further assume that (2.1) depends on the only small parameter $\sqrt{E}$.

The smallness of this parameter allows us to treat the flows in the bulk of the Earth's core as inviscid ones and neglect the last (viscid) term in rhs of (2.1). The neglecting of this term reduces the order of the equation and the no-slip boundary condition for it must respectively be replaced by others.

The Ekman suction boundary conditions are obtained by solution of the momentum and the induction equations into the boundary layer. The suction into the layer is proportional to the jump of the normal component of the curl of the flow in it. Respectively the boundary conditions for the flow out of the layer in the North and the South hemispheres take the form:

$$\begin{aligned} \bar{\rho}(r_1)\,\mathbf{r}_{\pm2}\cdot\mathbf{V}(\mathbf{r}_{\pm2}) &= \mp\sqrt{E}\,\gamma(\mathbf{r}_{\pm2})\,\mathbf{r}_{\pm2}\cdot\nabla\times\left[\mathbf{V}(\mathbf{r}_{\pm2}) - \mathbf{1}_\phi s\omega_2\right] \\ \bar{\rho}(r_1)\,\mathbf{r}_{\pm1}\cdot\mathbf{V}(\mathbf{r}_{\pm1}) &= \mp\sqrt{E}\,\gamma(\mathbf{r}_{\pm1})\,\mathbf{r}_{\pm1}\cdot\nabla\times\left[\mathbf{V}(\mathbf{r}_{\pm1}) - \mathbf{1}_\phi s\omega_1\right] \end{aligned} \quad (2.4)$$

where $\mathbf{r}_\pm = 1_s s \pm \mathbf{1}_z |z|$, $\omega_{1,2}$ is the angular velocity of the correspondent boundary and

$$\gamma(r_{1,2},\vartheta,\phi) = \bar{\rho}(r_{1,2})\sqrt{\frac{\sqrt{\cos^2\vartheta + B_r^4} - B_r^2}{2(\cos^2\vartheta + B_r^4}} \quad (2.5)$$

Equation (2.1) and the boundary conditions (2.4) depend on the small parameter $\sqrt{E}$ and they respectively hint that the flow can be searched in the form of expansion in this parameter:

$$\mathbf{V}(r) = \mathbf{V}^{(0)}(r) + \sqrt{E}\mathbf{V}^{(1)}(\mathbf{r}) + \ldots \qquad P(\mathbf{r}) = P^{(0)}(r) + \sqrt{E}P^{(1)}(\mathbf{r}) + \ldots \quad (2.6)$$

The substitution of (2.6) into (2.4) yields to the boundary condition of the leading

$$\bar{\rho}(\mathbf{r}_1)\,\mathbf{r}_{\pm1}\cdot\mathbf{V}^{(0)}(r_{\pm1}) = 0, \quad \bar{\rho}(\mathbf{r}_2)\,\mathbf{r}_{\pm2}\cdot\mathbf{V}^{(0)}(r_{\pm2}) = 0, \quad (2.7,8)$$

and the next

$$\bar{\rho}(r_1)\,\mathbf{r}_{\pm1}\cdot\mathbf{V}^{(1)}(\mathbf{r}_{\pm1}) = \pm\gamma(\mathbf{r}_{\pm1})\,\mathbf{r}_{\pm1}\cdot\nabla\times\mathbf{A}^{(0)}(\mathbf{r}_{\pm1}) \quad (2.9)$$

$$\bar{\rho}(r_2)\,\mathbf{r}_{\pm2}\cdot\mathbf{V}^{(1)}(\mathbf{r}_{\pm2}) = \pm\gamma(\mathbf{r}_{\pm2})\,\mathbf{r}_{\pm2}\cdot\nabla\times\mathbf{A}^{(0)}(\mathbf{r}_{\pm2}) \quad (2.10)$$

approximations, where

$$\mathbf{A}^{(0)}(\mathbf{r}_{\pm1,2}) = \mathbf{V}^{(0)}(r_{\pm1,2}) - \mathbf{1}_\phi s\omega_{1,2}. \quad (2.11)$$

Signs "+" and "–" at $\mathbf{r}_{\pm}$ refer to the North and the South hemispheres in the core. Instead of such presentation, we can search here for a solution in terms of values of the dipole and the quadruple symmetries in the upper (the North) hemisphere only. We will call a scalar field the quadruple one, if it is even in respect of z. Respectively, the field of the dipole type is an odd one in respect of z. We can say that the vector field is of a dipole type, if its *z* component is an even function of *z* and its *s* and ϕ components are odd functions. The vector field of the opposite symmetry, whose *z* component is an odd function of *z* and *s* and ϕ components are even functions, will be called the field of the quadruple symmetry.

Summarizing and subtracting "+" and "–" components of (2.7) and (2.8) we obtain the boundary conditions for the quadruple flow and the dipole flow of the leading approximation:

$$\bar{\rho}(r_1)\,\mathbf{r}_1\cdot\mathbf{V}_q^{(0)}(\mathbf{r}\ \ )=0\,,\quad \bar{\rho}(r_2)\,\mathbf{r}_2\cdot\mathbf{V}_q^{(0)}(\mathbf{r}\ \ )=0\,; \tag{2.7q, 8q}$$

$$\bar{\rho}(r_1)\,\mathbf{r}_1\cdot\mathbf{V}_d^{(0)}(\mathbf{r}\ \ )=0\,,\quad \bar{\rho}(r_2)\,\mathbf{r}_2\cdot\mathbf{V}_d^{(0)}(\mathbf{r}\ \ )=0\,. \tag{2.7d, 8d}$$

Proceeding analogically with (2.9) and (2.10) one can obtain the boundary conditions of the next approximation:

$$\bar{\rho}(r_1)\,\mathbf{r}_1\cdot\mathbf{V}_q^{(1)}(\mathbf{r}_1)=-\gamma_q(r_1)\,\mathbf{r}_1\cdot\nabla\times\mathbf{A}_q^{(0)}(\mathbf{r}_1)-\gamma_d(\mathbf{r}_1)\,\mathbf{r}_1\cdot\nabla\times\mathbf{A}_d^{(0)}(\mathbf{r}_1)\,; \tag{2.9q}$$

$$\bar{\rho}(r_2)\,\mathbf{r}_2\cdot\mathbf{V}_q^{(1)}(\mathbf{r}_2)=\gamma_q(r_2)\,\mathbf{r}_2\cdot\nabla\times\mathbf{A}_q^{(0)}(\mathbf{r}_2)+\gamma_d(\mathbf{r}_2)\,\mathbf{r}_2\cdot\nabla\times\mathbf{A}_d^{(0)}(\mathbf{r}_2)\,; \tag{2.10q}$$

$$\bar{\rho}(r_1)\,\mathbf{r}_1\cdot\mathbf{V}_d^{(1)}(\mathbf{r}_1)=-\gamma_q(r_1)\,\mathbf{r}_1\cdot\nabla\times\mathbf{A}_d^{(0)}(\mathbf{r}_1)-\gamma_d(\mathbf{r}_1)\,\mathbf{r}_1\cdot\nabla\times\mathbf{A}_q^{(0)}(\mathbf{r}_1)\,; \tag{2.9d}$$

$$\bar{\rho}(r_2)\,\mathbf{r}_2\cdot\mathbf{V}_d^{(1)}(\mathbf{r}_2)=\gamma_q(r_2)\,\mathbf{r}_2\cdot\nabla\times\mathbf{A}_d^{(0)}(\mathbf{r}_2)+\gamma_d(\mathbf{r}_2)\,\mathbf{r}_2\cdot\nabla\times\mathbf{A}_q^{(0)}(\mathbf{r}_2)\,; \tag{2.10d}$$

where

$$\mathbf{A}_q^{(0)}(\mathbf{r}_{1,2})=\mathbf{V}_q^{(0)}(\mathbf{r}_{1,2})-\mathbf{1}_\phi s\omega_{1,2}\,;\quad .\,\mathbf{A}_d^{(0)}(r_{1,2})=\mathbf{V}_d^{(0)}(\mathbf{r}_{1,2})\ . \tag{2.11q, d}$$

Finally, we can obtain the boundary condition for a particular but important case of the axisymmetric flow. This flow for both approximations has a form:

$$\bar{\mathbf{V}}(s,z)=-\frac{\mathbf{1}_\phi}{\bar{\rho}(r)s}\times\nabla\bar{\chi}(s,z)\,+\,\mathbf{1}_\phi\bar{V}_\phi \tag{2.12}$$

Combining (2.7,8) and (2.12) one can obtain: $\partial\bar{\chi}^{(0)}(r_{1,2},\vartheta)/\partial\vartheta=0$ where (r,ϑ,ϕ) are the polar spherical coordinates. From here it follows that the streamline function $\bar{\chi}^{(0)}(r,\vartheta)$ is constant on the boundaries. Further we will accept this constant to be zero:

$$\bar{\chi}^{(0)}(r_{1,2,}\vartheta)=0 \tag{2.13}$$

Eq. (2.12) also can be transformed to its quadruple-dipole form:

$$\overline{\mathbf{V}}_q(s,z) = -\frac{\mathbf{1}_\phi}{\bar\rho(r)s}\times\nabla\bar\chi_d + \mathbf{1}_\phi\bar{V}_{q\phi}, \qquad \overline{\mathbf{V}}_d(s,z) = -\frac{\mathbf{1}_\phi}{\bar\rho(r)s}\times\nabla\bar\chi_q + \mathbf{1}_\phi\bar{V}_{d\phi}; \qquad (2.12q,d)$$

Then averaging (2.9 q, d), (2.10 q,d) over ϕ we obtain the boundary conditions of the next approximation for the axisymmetric flow:

$$\frac{1}{\sin\vartheta}\frac{\partial\bar\chi_d^{(1)}(r_1,\vartheta)}{r_1\,\partial\vartheta} = -\overline{\gamma_d(\mathbf{r})\,\mathbf{r}_1\cdot\nabla\times\mathbf{A}_q^{(0)}(\mathbf{r})} - \overline{\gamma_d(\mathbf{r}_1)\,\mathbf{r}_1\cdot\nabla\times\mathbf{A}_d^{(0)}(\mathbf{r}_1)}; \qquad (2.14q)$$

$$\frac{1}{\sin\vartheta}\frac{\partial\bar\chi_d^{(1)}(r_2,\vartheta)}{r_2\,\partial\vartheta} = \overline{\gamma_q(\mathbf{r})\,\mathbf{r}_2\cdot\nabla\times\mathbf{A}_q^{(0)}(\mathbf{r})} + \overline{\gamma_d(\mathbf{r}_2)\,\mathbf{r}_2\cdot\nabla\times\mathbf{A}_d^{(0)}(\mathbf{r}_2)}; \qquad (2.15q)$$

$$\frac{1}{\sin\vartheta}\frac{\partial\bar\chi_q^{(1)}(r_1,\vartheta)}{r_1\,\partial\vartheta} = -\overline{\gamma_q(\mathbf{r})\,\mathbf{r}_1\cdot\nabla\times\mathbf{A}_d^{(0)}(\mathbf{r})} - \overline{\gamma_d(\mathbf{r}_1)\,\mathbf{r}_1\cdot\nabla\times\mathbf{A}_q^{(0)}(\mathbf{r}_1)}; \qquad (2.14d)$$

$$\frac{1}{\sin\vartheta}\frac{\partial\bar\chi_q^{(1)}(r_2,\vartheta)}{r_2\,\partial\vartheta} = \overline{\gamma_q(\mathbf{r})\,\mathbf{r}_2\cdot\nabla\times\mathbf{A}_d^{(0)}(\mathbf{r})} + \overline{\gamma_d(\mathbf{r}_2)\,\mathbf{r}_2\cdot\nabla\times\mathbf{A}_q^{(0)}(\mathbf{r}_2)} \qquad (2.15d)$$

Here the bar over a quantity means its ϕ averaging value.

Let us assume that both problems of the leading and the next approximations are resolved and a solution in the form (2.6) is obtained. As (2.6) shows, the leading part of the flow does not depend on the Ekman number. Respectively, the flow depends on the only parameter λ.

Thus we can conclude that in the case of small $\sqrt{E}$ and R_o the flow doesn't depend separately on both these parameters, but only on their relation $\lambda = \sqrt{E}/R_o$. It means that λ at small values of $\sqrt{E}$ and R_o becomes the universal parameter for the fast rotating flow as e.g. the Reynolds number in the hydrodynamics. We noticed in the beginning of this section that the Rossby number and the square root of the Ekman number are both of order of 10^{-5}. However the problem with $E \sim 10^{-5}$ is out of up-to-date computer possibilities. Nevertheless, this problem can be relatively easily solved for the greater values $\sqrt{E}$ and R_o e.g. $\sqrt{E} = R_o = 10^{-2}$. Our discussion above shows that both flows (with the same λ) must be close to one another. Respectively, a very complicated from computer simulation point of view problem can be replaced with another, more resolvable, one.

By replacing (2.6) into (2.1), we can obtain the momentum equations of the leading and the next approximations. However, prior to this, let us consider the equations for the kinetic energy and the angular momentum.

Kinetic energy

Multiplying (2.1) by **V** we obtain the equation for the kinetic energy in the bulk of the core:

$$\frac{1}{\lambda}\frac{\partial}{\partial t}\frac{\overline{\rho}\,\mathbf{V}^2}{2} = \frac{-\nabla\cdot P\,\overline{\rho}\,\mathbf{V} + \mathbf{V}\cdot\mathbf{F}}{\sqrt{\mathrm{E}}} \tag{3.1}$$

The replacement of (2.6) into (3.1) gives the equation for the kinetic energy of the leading approximation:

$$-\boldsymbol{\nabla}\cdot \mathrm{P}^{(0)}\,\overline{\rho}\,\mathbf{V}^{(0)} + \mathbf{V}^{(0)}\cdot\mathbf{F} = 0 \tag{3.2}$$

Equation (3.2) shows that in order to avoid the enormous changes of the kinetic energy in (3.1) the pressure of the leading approximation adjusts in such a way that its work compensates the work of the force at any location. Then the whole (integrated over the whole core) force work of the leading approximation vanishes due to the boundary conditions (2.7, 8):

$$\iiint \mathbf{V}^{(0)}\cdot\mathbf{F}\,\mathrm{d}^3\mathbf{r} = \iint \mathrm{P}^{(0)}\,\overline{\rho}(\mathrm{r})\,\mathbf{n}\cdot\mathbf{V}^{(0)}\,\mathrm{dS} = 0\,. \tag{3.3}$$

It follows from (3.3) that the whole Archimedean work of the leading approximation converts into the work of the Lorentz force (and then into the Ohmic dissipation).

Subtracting (3.2) from (3.1) we obtain the equation for the kinetic energy in the next approximation:

$$\frac{1}{\lambda}\frac{\partial}{\partial \mathrm{t}}\frac{\overline{\rho}\,\mathbf{V}^{(0)^2}}{2} = -\nabla\cdot\left[\mathrm{P}^{(0)}\overline{\rho}\,\mathbf{V}^{(1)} + \mathrm{P}^{(1)}\overline{\rho}\,\mathbf{V}^{(0)}\right] + \mathbf{V}^{(1)}\cdot\mathbf{F} \tag{3.4}$$

The small kinetic energy changes are defined by the terms of the next approximation. Though both (the leading and the next approximations) pressures are included into (3.4), the last one vanishes due to (2.7,8) from the equation for the whole kinetic energy:

$$\frac{\partial}{\partial \mathrm{t}}\iiint \mathrm{d}^3\mathrm{r}\,\frac{\overline{\rho}\,\mathbf{V}^{(0)^2}}{2} = \lambda\left[\iiint \mathrm{d}^3\mathbf{r}\,\mathbf{V}^{(1)}\cdot\mathbf{F} - \iint \mathrm{dS}\,\mathrm{P}^{(0)}\overline{\rho}\,\mathbf{n}\cdot\mathbf{V}^{(1)}\right] \tag{3.5}$$

Eq. (3.5) shows that the kinetic energy of the leading approximation flow changes due to the work of the force and the pressure of the leading approximation over the flow of the next approximation. The pressure work is transformed into energy flux in the boundary layer. Its value is defined by the Ekman suction (2.9,10). This flux compensates the energy loss into the layer.

Let us imagine that the liquid in the Earth's core was not conductive. Then the magnetic field and respectively the Lorentz force would be absent. The equation (3.3) in this case takes a form:

$$\iiint \mathbf{V}^{(0)}\cdot\mathbf{F}^{\mathrm{a}}\,\mathrm{d}^3\mathbf{r} = 0 \tag{3.6}$$

This means that the flow of the leading approximation establishes in such a way that the work of the Archimedean force is balanced by the heating, due to the opposite process.

Angular momentum

Averaging the ϕ component of the momentum equation (1.1) over ϕ we obtain the equation for the density of the angular momentum relative axis OZ:

$$\frac{1}{\lambda}\frac{\partial s\bar{\rho}\,\bar{V}_\phi}{\partial t} = \nabla\cdot\overline{\mathbf{m}}, \quad \text{where} \quad \overline{\mathbf{m}} = \frac{s\,\overline{\mathbf{B}_m B_\phi} + \mathbf{1}_z\bar{\chi}}{\sqrt{E}} - \frac{\bar{\rho}}{\lambda}s\,\overline{\mathbf{V}_m V_\phi} \quad (+\sqrt{E}\,\nabla\, s\,\bar{\rho}\,\bar{V}_\phi) \quad (4.1)$$

Here index "m" refers to the meridional (s and z or r and ϑ) components of any vector and $\bar{\chi}(s,z)$ is the streamline function of the axisymmetric (2.12) part of the flow. Eq. (4.1) is, in fact, the law of conservation of the angular momentum relative axis OZ. This means that the changes of this momentum in any axisymmetric volume are defined only by the momentum flux $\overline{\mathbf{m}}$ across its boundaries. This flux, as (4.1) shows, consists of three parts. The first of them, $s\,\overline{\mathbf{B}_m B_\phi}/\sqrt{E}$, is related to the torque created by the Lorentz force. The second one, $\mathbf{1}_z\bar{\chi}/\sqrt{E}$, describes the transportation in s direction of the angular momentum connected with the angular velocity $\mathbf{1}_z\Omega$ of the reference framework. And, finally the last term of the momentum flux $\overline{\mathbf{m}}$ in (4.1), $\lambda\bar{\rho}s\,\overline{\mathbf{V}_m V_\phi}$, is simply the flux of the angular momentum density $\bar{\rho}s V_\phi$ in direction of the meridional flow $\mathbf{V}_m$ averaged over ϕ. Since Ω » V_ϕ/s, the relation between the third and the second terms is small ~ R_o. If neglecting this term, we obtain $\overline{\mathbf{m}}$ in the leading approximation: $\sqrt{E}\,\overline{\mathbf{m}}^{(0)}(s,z) = s\,\overline{\mathbf{B}_m B_\phi} - \mathbf{1}_z\bar{\chi}^{(0)}$. Respectively, the leading approximation of (4.1) yields:

$$\sqrt{E}\,\nabla\cdot\overline{\mathbf{m}}^{(0)} = \frac{\partial\bar{\chi}^{(0)}(s,z)}{\partial z} + \nabla\cdot s\,\overline{\mathbf{B}_m B_\phi} = 0 \quad (4.2)$$

Equation (4.2) with boundary condition (2.12) defines the meridional flow of the leading approximation. Its streamline function can be easily expressed in terms of integrals in respect of z. Integration of (4.2) over z between two solid boundaries yields to Taylor constraint:

$$\int_{z_{botton}}^{z_{top}} \nabla.s\overline{\mathbf{B}_m B_\phi} = \left(=\int_{z_{botton}}^{z_{top}} \bar{F}_\phi\right) = \bar{\chi}^{(0)}(s,z_{botton}) - \bar{\chi}^{(0)}(s,z_{top}) = O(\sqrt{E}) \quad (4.3)$$

Thus Taylor constraint becomes the required condition for the existence of the axisymmetric meridional flow.

Let us integrate (4.2) over the whole volume of the liquid core. Since $\bar{\chi}^{(0)}(r_{1,2}) = 0$ we obtain:

$$\int_0^\pi \sin^2\vartheta\left[r_1^3\,\overline{B_r B_\phi}(r_1,\vartheta) - r_2^3\,\overline{B_r B_\phi}(r_2,\vartheta)\right]d\vartheta = O(\sqrt{E}) \quad (4.4)$$

This estimation is interesting to compare to the note of Glatzmaier and Roberts in (1996b) who write: "*... That is, when we integrate the absolute value of the moment of the magnetic* *stress* $\left|B_r B_\phi\right|/\mu_o$ *(instead of* $B_r B_\phi/\mu$*) over the inner core boundary, we consistently obtain values three orders of magnitude greater than* Γ_B *".*

Combining (2.6) and (4.1) and (4.2) shows that the change of the angular momentum is defined by the momentum flux of the next approximation:

$$\frac{\partial s\overline{\rho}\,\overline{V}_\phi^{(0)}}{\partial t} = \nabla\cdot\left[\mathbf{1}_z\lambda\,\overline{\chi}^{(1)} - \overline{\mathbf{V}_m^{(0)}\overline{\rho}\,s\,V_\phi^{(0)}}\right]. \quad (4.5)$$

This equation allows studying the distribution of the angular momentum in the liquid core. Let us consider the part of the liquid core bounded by a cylinder with radius *s*. The angular momentum of this volume can be obtained by the integration of (4.5) over it. Taken the divergence theorem into account we obtain:

$$\frac{dM(s)}{dt} = \lambda\Upsilon_\delta(s) + \Upsilon_F(s) \quad (4.6)$$

where

$$M(s) = \int_0^s s\,ds\int_{z_b}^{z_2}\overline{\rho}\,s\,\overline{V}_{q\phi}^{(0)}(s,z)\,dz \quad (4.7)$$

is the angular momentum of the volume and

$$z_b = z_1 = \sqrt{r_1^2 - s^2} \quad \text{when} \quad s \le r_1 \quad \text{and} \quad z_b = 0 \quad \text{when} \quad s \ge r_1\,. \quad (4.8)$$

The torque Υ_δ acting on the boundaries and the torque Υ_F acting on the cylinder surface are defined by the expressions:

$$\Upsilon_\delta(s) = \int_0^s s\,ds\,\Pi_d(s)\,, \quad (4.9)$$

$$\Pi_{d,q}(s) = \overline{\chi}_{d,q}^{(1)}(s,z_2) - \overline{\chi}_{d,q}^{(1)}(s,z_1)\,,\ \text{if } s < r_1 \text{ and } \Pi_{d,q}(s) = \overline{\chi}_{d,q}^{(1)}(s,z_2) \text{ if } s > r_1 \quad (4.9a)$$

$$\Upsilon_F(s) = s^2\int_{z_b}^{z_2}dz\,\overline{\rho}\left[\overline{V_{qs}^{(0)}\,V_{q\phi}^{(0)}}(s,z) + \overline{V_{ds}^{(0)}\,V_{d\phi}^{(0)}}(s,z)\right]. \quad (4.10)$$

Thus the Earth's core angular momentum distribution *M(s)* is defined by two torques. The first of them $\Upsilon_\delta(s)$ is the torque connected to the boundary flux on both boundaries. This change of the momentum is related to the transport of the angular momentum connected to the rotation of the reference framework with the angular velocity $\mathbf{1}_z\Omega$. Function $\overline{\chi}^{(1)}(s,z)$ can be presented as a sum $\overline{\chi}^{(1)}(s,z) = \overline{\chi}_d^{(1)}(s,z) + \overline{\chi}_q^{(1)}(s,z)$ of its odd and even components in respect of *z*. The transfer carried out by the flux related to $\overline{\chi}_q^{(1)}$ is compensated in the North and the South hemispheres. Therefore, only the transport connected to the odd function $\overline{\chi}_d^{(1)}$, changes the angular momentum. For example $\overline{\chi}^{(1)} = \overline{\chi}^{(1)}(r)(2sz/r^2)$ is the flux that changes this momentum effectively.

The second torque $\Upsilon_F(s)$ defines the flow velocity which is determined in terms of the force into the liquid core. Therefore we call it the force torque. It is defined by the even component of $\overline{V_s^{(0)}V_\varphi^{(0)}}$ in respect of *z*. So the flows of the dipole and the quadruple types generate this torque effectively. The force torque works in the whole core but as (4.10) shows, it vanishes at $s = r_2$. So the equation for the angular momentum of the whole liquid core converts into

$$\frac{dM_2}{dt} = \lambda\left[\frac{r_1^2}{2}\int_0^\pi d\vartheta\ \sin 2\vartheta\ \overline{\chi}_d^{(1)}(r_1,\vartheta) - \frac{r_2^2}{2}\int_0^\pi d\vartheta \sin 2\vartheta\, \overline{\chi}_d^{(1)}(r_2,\vartheta)\right] \tag{4.11}$$

where $M_2 = M(r_2)$.

Taking (2.12) into account we can rewrite (4.11) also in the form:

$$\frac{dM_2(t)}{dt} = \lambda\ \frac{r_2^4}{4}\int_0^\pi d\vartheta\ \sin\vartheta\ \cos 2\vartheta\left[\bar{\rho}(r_2)\,\overline{V}_r^{(1)}(r_2,\vartheta) - \frac{r_1^4}{r_2^4}\bar{\rho}(r_1)\ \overline{V}_r^{(1)}(r_1,\vartheta)\right] \tag{4.12}$$

Note that as (4.11) shows, the angular momentum of the liquid core is driven by two torques acting on ICB and CMB. Respectively, the same torqueses (with the opposite signs) act over the inner core and the mantle. This yields to equations for their angular momentum:

$$\frac{d\omega_1}{dt} = -\frac{\lambda}{I_1}\frac{r_1^2}{2}\int_0^\pi d\vartheta \sin 2\vartheta\, \overline{\chi}_d^{(1)}(r_1,\vartheta) \tag{4.13}$$

$$\frac{d\omega_2}{dt} = -\frac{\lambda}{I_3}\frac{r_2^2}{2}\int_0^\pi d\vartheta \sin 2\vartheta\, \overline{\chi}_d^{(1)}(r_2,\vartheta), \tag{4.14}$$

where I_1 and I_3 are the inertia momentum of the inner core and the mantle, respectively.

By assuming that the mass flux on both boundaries is of the same order, we obtain a crude estimation for the relation between the angular accelerations of the inner core and the mantle:

$$\frac{d\omega_1}{dt}\Big/\frac{d\omega_2}{dt} \sim \frac{\rho_3}{\rho_1}\left(\frac{r_3}{r_2}\right)^2\left(\frac{r_3}{r_1}\right)^3 \sim 7\times 10^1, \tag{4.15}$$

where ρ_1 and ρ_3 are densities of the inner core and the mantle and r_3 is the radius of the Earth.

The leading approximation problem

The substitution of (2.6) into (2.1) yields to the equations of the leading approximation:

$$-\bar{\rho}\,\nabla P^{(0)} - \bar{\rho}\,\mathbf{1}_z \times \mathbf{V}^{(0)} + \mathbf{F} = 0, \qquad \nabla\cdot\bar{\rho}\mathbf{V}^{(0)} = 0 \tag{5.1,2}$$

Equation (5.1) is obtained in supposition that the typical time scale t_1 of the processes is of order of L_1/V_1. This supposition fails for the very fast processes when $(V_1/L_1)t_1\lambda \sim \sqrt{E}$. The typical time for this fail is very small: $t_1 \sim (L_1/V_1)(\sqrt{E}/\lambda) = = (1/2\Omega)$. It follows from here that equation (5.1) is valid if the typical time of the correspondent processes is much larger than a few hours: t_1 » $(1/2\Omega)$.

The solution of the equations (5.1,2) obeys to the no-normal-flow boundary conditions (2.7,8) of the leading approximation. Let us first discuss the non-axisymmetric flow, $\tilde{\mathbf{V}}^{(0)}(r)$. Using (5.1,2), this flow can be expressed in terms of the force $\tilde{\mathbf{F}}$. The ambiguity in the meridional components of this flow can be removed by the boundary conditions (2.7,8). Then the non-axisymmetric part of the azimuthal flow $\mathbf{1}_\phi \tilde{V}_\phi^{(0)}$ also can be defined from the continuity equation (5.2). Therefore the non-axisymmetric flow of the leading approximation is wholly defined by the problem (5.1,2), (2.7,8).
The meridional part of the axisymmetric flow, $\overline{\mathbf{V}}^{(0)}(s,z) = -(\mathbf{1}_\phi/\overline{\rho}(r)s)\times\nabla\overline{\chi}^{(0)} + \mathbf{1}_\phi\overline{V}_\phi^{(0)}$, is also defined by (5.1,2), (2.7,8). The stream line function obeys the equation (4.2) and the boundary conditions (2.13). These two conditions for the only function $\overline{\chi}^{(0)}(s,z)$ are over-constraint for the problem. So an additional condition (4.3) has to be satisfied in the liquid core. Thus the Taylor constraint plays the role of the necessary condition to define the axisymmetric meridional flow.

The azimuthal component of the axisymmetric flow is not included into the boundary conditions (2.7,8). Respectively it cannot be determined from them. It satisfies the continuity equation identically: $\nabla\cdot\overline{\rho}(r)\mathbf{1}_\phi\overline{V}_\phi^{(0)}(s,z) \equiv 0$ and therefore cannot be determined by this equation either. Therefore the azimuthal axisymmetric flow of the leading approximation cannot be determined in its frame. The next approximation is needed to define it. But nevertheless an essential part of the axisymmetric azimuthal flow can be defined in this approximation.

Dividing (5.1) by $\overline{\rho}$ and applying the curl operator yields to

$$\frac{\partial \mathbf{V}^{(0)}}{\partial z} = \mathbf{1}_z(\nabla\cdot\mathbf{V}^{(0)}) - \nabla\times\frac{\mathbf{F}}{\overline{\rho}} \qquad (5.3)$$

It follows from here that the ϕ component of the axisymmetric azimuthal flow can presented as a sum of the force, $\overline{V}^F = s\overline{\omega}^F(s,z)$, and the geostrophic, $\overline{V}^g = s\overline{\omega}^g(s)$, flows: $\overline{V}_\phi^{(0)} = s\overline{\omega}^F + s\overline{\omega}^g$, where the equation for $\overline{\omega}^F$ takes a form:

$$\frac{\partial\overline{\omega}^F}{\partial z} = -\frac{1_\phi}{s}\nabla\times\frac{\overline{\mathbf{F}}}{\overline{\rho}}; \quad \left(= \nabla\cdot\left[\frac{\mathbf{1}_\phi}{s}\times\frac{\overline{\mathbf{F}}}{\overline{\rho}}\right]\right) \qquad (5.4)$$

The force in (5.4) consists of the Archimedean, $\overline{\mathbf{F}}^a = \mathbf{r}A(r)\overline{\rho}\overline{C}(s,z)$, and the Lorentz, $\mathbf{F}^b = R_m\mathbf{J}\times\mathbf{B}$, parts. Respectively, the axisymmetric force azimuthal flow can be presented as a sum of the Archimedean and the magnetic ones: $\overline{\omega}^{F}(s,z)=\overline{\omega}^{a}(s,z)+\overline{\omega}^{b}(s,z)$. The equations for both flows can be obtained directly from (5.4):

$$\frac{\partial\overline{\omega}^{a}}{\partial z}=\frac{A(r)}{r\sin\vartheta}\frac{\partial\overline{C}(r,\vartheta)}{\partial\vartheta} \tag{5.5}$$

$$\frac{\partial\overline{\omega}^{b}}{\partial z}=R_{m}\boldsymbol{\nabla}\cdot\left[\frac{\overline{\mathbf{J}B_{\phi}-\mathbf{B}J_{\phi}}}{s\overline{\rho}}\right]. \tag{5.6}$$

Thus $\overline{V}_{\phi}^{(0)}(s,z)$ consists of two parts: the force flow, $\overline{V}_{\phi}^{F}$, depending on *s* and *z* and the geostrophic one, $\overline{V}_{\phi}^{g}(s)$, depending on *s* only. The first of them is defined by (5.4), but the second one must be determined in the next approximation.

In conclusion of this section let us emphasize again that all the components of the flow except its geostrophic part can be defined in the leading approximation. Respectively the flow of the leading approximation can be presented as a sum of the force flow $\mathbf{V}^{F}(s,\phi,z)$ which is defined in this approximation and the unknown geostrophic flow $\mathbf{1}_{\phi}V^{g}=\mathbf{1}_{\phi}s\overline{\omega}^{g}(t,s)$:

$$\mathbf{V}^{(0)}=\mathbf{V}^{F}(s,\phi,z)+\mathbf{1}_{\phi}s\,\overline{\omega}^{g}(t,s) \tag{5.7}$$

The force flow $\mathbf{V}^{F}$ consists of the axisymmetric and the non-axisymmetric parts:

$$\tilde{V}^{(0)}=\tilde{\mathbf{V}}^{F}(s,\phi,z)\,;\qquad \overline{\mathbf{V}}^{F}(s,z)=-\frac{\mathbf{1}_{\phi}}{\overline{\rho}(r)s}\times\nabla\overline{\chi}^{F}(s,z)+\mathbf{1}_{\phi}\overline{V}_{\phi}^{F}(s,z)\,, \tag{5.8}$$

The axisymmetric part of the meridional force flow can be obtained by integration of (4.2) under the boundary conditions (2.13). Its azimuthal part is defined by the equation (5.4). The non-axisymmetric part of the force flow can be analytically obtained from the equations (5.1,2) and the boundary conditions (2.7,8). This problem also can be solved numerically.

The force flow establishes instantly by means of pressure. Respectively, its time dependence is defined by the time dependence of the force. Thus it has no its own time dependence. To emphasize this, we do not include *t* in the list of its variables.

The next approximation problem

In order to obtain the equations of the next approximation, we substitute (2.6) into (2.1,2) and take (5.1,2) into account. Then we get:

$$\frac{\partial\mathbf{V}^{(0)}}{\partial t}-\mathbf{V}^{(0)}\times\nabla\times\mathbf{V}^{(0)}=-\lambda\left[\boldsymbol{\nabla}P^{(1)}+\mathbf{1}_{z}\times\mathbf{V}^{(1)}\right];\quad \nabla\cdot\overline{\rho}\mathbf{V}^{(1)}=0\,. \tag{6.1,2}$$

Note that, since the force flow does not depend directly on time, the only nonvanishing time derivative in lhs of (6.1) is that of the geostrophic velocity. Equations (6.1,2) could be solved together with the boundary conditions (2.9,10) to find the geostrophic flow of the leading order approximation and the flow of the next order approximation. However, we will see later that the geostrophic flow interacts only with the boundary flux. So we need the only equation which connects the geostrophic velocity and the streamline function of the next approximation. This equation which is the ϕ component of (6.1), can be directly obtained from (4.5), taking (5.7) into account.

$$\frac{\partial s\overline{\rho}\,\overline{V}_{\phi}^{g}(s,t)}{\partial t} = \nabla\cdot\left[\mathbf{1}_{z}\lambda\overline{\chi}^{(1)} - \overline{\mathbf{V}_{m}^{(0)}\overline{\rho}sV_{\phi}^{(0)}}\right] \tag{6.3}$$

In section 4 we have introduced the function M(s) describing the distribution of the angular momentum of the liquid core. We could consider also distributions of the angular momentums in the North $M_{+}(s)$ and in the South $M_{-}(s)$ hemispheres and respectively their quadruple $M_{q}(s)$ and dipole $M_{d}(s)$ components. The integration of (6.3) over the same volume as in section 4, gives the equations for $M_{q,d}(s)$:

$$\frac{dM_{q,d}(s)}{dt} = \lambda\Upsilon_{q,d\delta}(s) + \Upsilon_{q,dF}(s) \tag{6.4}$$

Here the quadruple and dipole angular components of the angular momentum are defined by the expressions:

$$M_{q,d}(s) = \int_{0}^{s} sds\, s^{2}\mathcal{R}(s)\,\overline{\omega}_{q,d}^{g}(s)\,dz\ , \quad \text{where} \quad \mathcal{R}(s) = \int_{z_b}^{z_2} dz\overline{\rho}(r) \tag{6.5}$$

Eq. (6.5) shows that the angular momentum of the whole body is presented as a sum of momentums of the liquid cylindrical shells rotating with the angular velocities $\overline{\omega}_{q,d}^{g}(s)$. The value $s^{2}R(s)$ plays the role of the inertia moments of the shells. Since $\overline{\omega}_{d}^{g}(s)$ vanishes outside the tangent cylinder, $M_{d}(s)$ vanishes here as well.

The torques $\Upsilon_{q,d\delta}$ and $\Upsilon_{q,dF}$ in (6.4) has a form like (4.9) and (4.10)

$$\Upsilon_{q,d\delta}(s) = \int_{0}^{s} s\,ds\,\Pi_{d,q}(s) \tag{6.6}$$

$$\Upsilon_{q,dF}(s) = s^{2}\int_{z_b}^{z_2} dz\,\overline{\rho}\left[\overline{V_{s}^{F}\,V_{\phi}^{F}(s,z)}\right]_{q,d} \tag{6.7}$$

where $\Pi_{d,q}(s)$ is defined by (4.9a) and z_{b} is defined again by (4.8). Since the whole flux across the cylinder surface between the ridged boundaries vanishes, the integral

$$\int dz\,\overline{\rho}\,s\,\overline{V}_{s}^{(0)}(s,z)\,V^{g}(s) = 0$$

vanishes as well. Respectively the geostrophic velocity is not included in rhs of (6.7).

By differentiating (6.4) in respect of *s* inside the tangent cylinder one can obtain equations for the geostrophic angular velocity:

$$\frac{\partial \omega_q^g(s)}{\partial t} - \frac{\lambda}{s^2} \frac{\overline{\chi}_d^{(1)}(s,z_2) - \overline{\chi}_d^{(1)}(s,z_1)}{\mathfrak{R}(s)} = \frac{\Im_q(s)}{\mathfrak{R}(s)}; \qquad s < r_1, \qquad (6.8)$$

$$\frac{\partial \omega_d^g(s)}{\partial t} - \frac{\lambda}{s^2} \frac{\overline{\chi}_q^{(1)}(s,z_2) - \overline{\chi}_q^{(1)}(s,z_1)}{\mathfrak{R}(s)} = \frac{\Im_d(s)}{\mathfrak{R}(s)}; \qquad s < r_1, \qquad (6.9)$$

where

$$\Im(s) = \frac{\overline{\rho}(r_1)\, \overline{V_s^F\, V_\phi^F}(s,z_1)}{z_1} - \frac{\overline{\rho}(r_2)\, \overline{V_s^F\, V_\phi^F}(s,z_2)}{z_2} + \frac{2}{s^2}\int_{z_1}^{z_2} dz\, \overline{\rho}\, \overline{V_s^F\, V_\phi^F}(s,z) \qquad s < r_1 \qquad (6.10)$$

Equations for the boundary fluxes $\overline{\chi}_{q,d}^{(1)}(s, r_{1,2})$ can be obtained from the boundary conditions (2.14 q,d), (2.15q,d):

$$\frac{\partial \overline{\chi}_d^{(1)}(s,r_1)}{\partial s} + \overline{\gamma}_q(\mathbf{r}_1)\frac{\partial s^2(\omega_q^g - \omega_1)}{\partial s} + \overline{\gamma}_d(\mathbf{r}_1)\frac{\partial s^2 \omega_d^g}{\partial s} = -W_d(s,r_1), \quad s < r_1, \qquad (6.11)$$

$$\frac{\partial \overline{\chi}_d^{(1)}(s,r_2)}{\partial s} - \overline{\gamma}_q(\mathbf{r}_2)\frac{\partial s^2(\omega_q^g - \omega_2)}{\partial s} - \overline{\gamma}_d(\mathbf{r}_2)\frac{\partial s^2 \omega_d^g}{\partial s} = W_d(s,r_2), \quad s < r_1, \qquad (6.12)$$

$$\frac{\partial \overline{\chi}_q^{(1)}(s,r_1)}{\partial s} + \overline{\gamma}_d(\mathbf{r}\,)\frac{\partial s^2(\omega_q^g - \omega_1)}{\partial s} + \overline{\gamma}_q(\mathbf{r}_1)\frac{\partial s^2 \omega_d^g}{\partial s} = -W_q(s,r_1), \quad s < r_1, \qquad (6.13)$$

$$\frac{\partial \overline{\chi}_q^{(1)}(s,r_2)}{\partial s} - \overline{\gamma}_d(\mathbf{r}_2)\frac{\partial s^2(\omega_q^g - \omega_2)}{\partial s} - \overline{\gamma}_q(\mathbf{r}_2)\frac{\partial s^2 \omega_d^g}{\partial s} = W_q(s,r_2), \quad s < r_1, \qquad (6.14)$$

where

$$W_q(s,r) = \frac{s}{z}\left[\overline{\gamma_q(\mathbf{r})\, \mathbf{r}\cdot\nabla\times\mathbf{V}_d^F(\mathbf{r})} + \overline{\gamma_d(\mathbf{r})\, \mathbf{r}\cdot\nabla\times\mathbf{V}_q^F(\mathbf{r})}\right] \qquad (6.15q)$$

$$W_d(s,r) = \frac{s}{z}\left[\overline{\gamma_q(\mathbf{r})\mathbf{r}\cdot\nabla\times\mathbf{V}_q^F(\mathbf{r})} + \overline{\gamma_d(\mathbf{r})\, \mathbf{r}\cdot\nabla\times\mathbf{V}_d^F(\mathbf{r})}\right] \qquad (6.15d)$$

Six equations (6.8,9) and (6.11-14) form the system of equations for the six quantities $\omega_{q,d}^g(s)$ and $\overline{\chi}_{q,d}^{(1)}(s, r_{1,2})$ inside the tangent cylinder. The angular momentum $M_{q,d}$ and respectively the geostrophic flow change due to the small fluxes of the angular momentum into the "cylinder" with radius *s*. That is why both values gradually change with time. The boundary fluxes restricted by the mass conservation law are controlled by the pressure acting on the time scale of order of few minutes. That is why the equations (6.11-14) do not depend on time.

The solution of these equations has to obey some additional conditions. The first of them arises due to the requirement of limitation of the *z* component of the flux at the axis OZ: $V_z^{(1)} = (1/s)(\partial\overline{\chi}^{(1)}/\partial s)$. The second one is a sequence of oddness of $\overline{\chi}_d^{(1)}$ in respect of

z. From here it follows that $\overline{\chi}_d^{(1)}$ has to vanish at the equator of ICB and CMB. By summarizing these conditions we obtain:

$$\overline{\chi}_{q,d}^{(1)}(s, r_{1,2}) = O(s^2) \quad \text{at } s = 0, \qquad \overline{\chi}_d^{(1)}(s, r_{1,2}) = 0 \quad \text{at } s = r_{1,2} \tag{6.16}$$

Mind that equations (6.8,9) and (6.11-14) have been obtained for the region inside the tangent cylinder. Outside it the dipole component of the geostrophic flow is absent. Therefore we need only one equation such as (6.8):

$$\frac{\partial \omega^g(s)}{\partial t} - \frac{\lambda}{s^2}\frac{\overline{\chi}_d^{(1)}(s, z_2)}{R(s)} = \frac{\Im_q(s)}{\Re(s)}\,, \qquad s > r_1, \tag{6.17}$$

where

$$\Im(s) = -\frac{\overline{V_s^F\, V_\phi^F}(s, z_2)}{z_2} + \frac{2}{s^2}\int_0^{z_2} dz\overline{\rho}\,\overline{V_s^F\, V_\phi^F}(s, z)\,, \quad s > r_1. \tag{6.18}$$

Eq, (6.17) includes the only boundary flux $\overline{\chi}_d^{(1)}(s, r_2)$. So in this region we need an equation only for this component of the flux:

$$\frac{\partial \overline{\chi}_d^{(1)}(s, r_2)}{\partial s} - \overline{\gamma}_q(\mathbf{r}_2)\frac{\partial s^2(\omega^g - \omega_2)}{\partial s} = W_d(s, r_2)\,, \quad s > r_1 \tag{6.19}$$

Thus outside the tangent cylinder we need only two equations (6.17,19) instead of six equations in the region inside it.

Note that equations (6.11-14) and (6.19) include two unknown values: the angular velocities of the inner core and the mantle. Therefore two additional equations (4.13,14) from section 4 must be added to complete the problem.

In conclusion of this section let us note that the problem is simplified drastically in the model excluding the inner core. Then it is reduced to the equations (6.17), (6.19) and (4.14) with conditions

$$\overline{\chi}_d^{(1)}(s, r_2) = O(s^2) \quad \text{at } s = 0; \qquad \overline{\chi}_d^{(1)}(s, r_2) = 0 \quad \text{at } s = r_2 \tag{6.20}$$

arising from (6.16).

Below we will discuss some other simple cases of the flow in the inner core of the Earth.

Quadruple flow

If the magnetic field is of the dipole symmetry and the temperature is an even function of z, then the flow can be a pure quadruple one. Expressions (2.12 q,d) show that $\overline{\chi}_q$ and ω_d^g vanish in this case. Therefore, if we omit in this section the indexes "d" and "q" at the remaining functions $\overline{\chi}_d$ and ω_q^g, this cannot lead to an ambiguity. It follows from (6.10) that $F_d = 0$. Therefore (6.9) is satisfied identically and (6.8) and (6.17) keep their form:

$$\frac{\partial \omega^g(s)}{\partial t} - \frac{\lambda}{s^2} \frac{\overline{\chi}^{(1)}(s,z_2) - \overline{\chi}^{(1)}(s,z_1)}{\mathfrak{R}(s)} = \frac{F_q(s)}{\mathfrak{R}(s)}; \qquad s < r_1 \qquad (7.1)$$

$$\frac{\partial \omega^g(s)}{\partial t} - \frac{\lambda}{s^2} \frac{\overline{\chi}^{(1)}(s,z_2)}{\mathfrak{R}(s)} = \frac{F_q(s)}{\mathfrak{R}(s)} \qquad s > r_1, \qquad (7.2)$$

where

$$\Im_q = \frac{\overline{\rho}(r_1)\,\overline{V^F_{qs}\,V^F_{q\phi}}(s,z_1)}{z_1} - \frac{\overline{\rho}(r_2)\,\overline{V^F_{qs}\,V^F_{q\phi}}(s,z_2)}{z_2} + 2\int_{z_1}^{z_2} dz\,\overline{\rho}\,\frac{\overline{V^F_{qs}\,V^F_{q\phi}}(s,z)}{s^2} \qquad s < r_1$$

$$\Im_q(s) = -\frac{\overline{V^F_{qs}\,V^F_{q\phi}}(s,z_2)}{z_2} + \frac{2}{s^2}\int_{z_1}^{z_2} dz\,\overline{\rho}\,\overline{V^F_{qs}\,V^F_{q\phi}}(s,z) \qquad s > r_1 \qquad (7.3)$$

It follows from (2.5) that $\overline{\gamma}_d$ vanishes when the magnetic field is of a dipole type. As (6.15) show, W_q vanishes and W_d takes a simple form: $W_d(s,r) = (s/z)\,\overline{\gamma_q(\mathbf{r})\mathbf{r}\cdot\nabla\times\mathbf{V}^F_q(\mathbf{r})}$. Then (6.13,14) are satisfied identically and (6.11,12) transform into

$$\frac{\partial \overline{\chi}^{(1)}(s,r_1)}{\partial s} + \overline{\gamma}_q(\mathbf{r})\frac{\partial s^2(\omega^g - \omega_1)}{\partial s} = -\frac{s}{z_1}\overline{\gamma_q(\mathbf{r}_1)\mathbf{r}.\nabla\times\mathbf{V}^F_q(\mathbf{r}_1)}\,, \qquad s < r_1 \qquad (7.4)$$

$$\frac{\partial \overline{\chi}^{(1)}(s,r_2)}{\partial s} + \overline{\gamma}_q(\mathbf{r})\frac{\partial s^2(\omega^g - \omega_2)}{\partial s} = -\frac{s}{z_2}\overline{\gamma_q(\mathbf{r}_2)\mathbf{r}.\nabla\times\mathbf{V}^F_q(\mathbf{r}_2)}\,, \qquad s < r_2 \qquad (7.5)$$

Thus in the case of quadruple flow, the number of the unknown variables reduces to three ones: $\overline{\chi}^{(1)}(s,r_2)$, $\overline{\chi}^{(1)}(s,r_1)$ and $\omega^g(s)$. Inside the tangent cylinder the equations for them are (7.1), (7.4) and (7.5). Outside the tangent cylinder we have only two unknown variables $\overline{\chi}^{(1)}(s,r_2)$, $\overline{\chi}^{(1)}(s,r_1)$ and $\omega^g(s)$ which have to be defined from (7.2) and (7.5). All the solutions have to satisfy the conditions (6.16). The angular velocities ω_1 and ω_2 of the inner core and the mantle are defined by the equations (4.13,14).

In the case of the axisymmetric magnetic field γ_q does not depend on ϕ. Taking (5.4) into account we obtain W_d:

$$W_d(s,r) = \overline{\gamma}_q\left[\frac{\partial s^2 \overline{\omega}^F_q}{\partial s} + \frac{s^2}{z}\mathbf{1}_\phi\cdot\nabla\times\frac{\overline{\mathbf{F}}_q}{\overline{\rho}}\right]$$

Then equations (7.4,5) are simplified:

$$\frac{\partial \overline{\chi}^{(1)}(s,r_1)}{\partial s} + \overline{\gamma}_q(s,r_1)\frac{\partial}{\partial s}s^2\left[\omega^g(s) + \omega^F_q(s,r_1) - \omega_1\right] = -\overline{\gamma}_q(s,r_1)\frac{s^2}{z_1}\mathbf{1}_\phi\cdot\nabla\times\frac{\overline{\mathbf{F}}_q}{\overline{\rho}}(s,r_1); \qquad (7.6)$$

$$\frac{\partial \overline{\chi}^{(1)}(s,r_2)}{\partial s}+\overline{\gamma}_q(s,r_2)\frac{\partial}{\partial s}s^2\left[\omega^g(s)+\omega_q^F(s,r_2)-\omega_2\right]=\overline{\gamma}_q(s,r_2)\frac{s^2}{z_2}\mathbf{1}_\phi\cdot\nabla\times\frac{\overline{\mathbf{F}}_q}{\overline{\rho}}(s,r_2). \qquad (7.7)$$

Inviscid flow

The estimate (2.3) shows that for the molecular viscosity $\nu=\nu_M$ the parameter λ is a relatively small one: $\lambda \sim 10^{-2}$. That is why the consideration of the problem with a small λ is a problem of physical interest.

As (6.8), (6.9) and (6.17) show, the geostrophic flow changes due to the difference of the meridional boundary fluxes on the top and the bottom boundaries and due to the angular momentum connected with the azimuthal force velocity, $s\overline{\rho}\overline{V}_\phi^F$, transferred by the meridional force flow, $\overline{V}_s^F$. It can be assumed, that in the inviscid limit, when $\lambda \to 0$, the first of these mechanisms attenuates and respectively these equations transform into the following ones:

$$\frac{\partial \omega_q^g(s)}{\partial t}=\frac{\Im_q(s)}{\Re(s)}, \qquad s<r_1; \qquad (8.1)$$

$$\frac{\partial \omega_d^g(s)}{\partial t}=\frac{\Im_d(s)}{\Re(s)}, \qquad s<r_1; \qquad (8.2)$$

$$\frac{\partial \omega^g(s)}{\partial t}=\frac{\Im_q(s)}{\Re(s)}, \qquad s>r_1; \qquad (8.3)$$

where $\Im(s)$ is defined by (6.10) in terms of the force flow. We declared in section 5 that the force flow is determined completely by the forces. The following scheme of solving the geodynamo problem on this base can be imagined. First, we can determine the temperature and the magnetic field distributions at a given time step. Then we obtain the force flow which establishes instantly by means of pressure. This flow defines rhs of (8.1-3). By integrating these equations we obtain new values of the geostrophic flow which have to be used in the next time step (together with the force flow, of course).

Note that as (3.5) and (4.11) show, the geostrophic flow in the inviscid limit evolves in such a way that the kinetic energy and the angular momentum do not change. The angular velocities of the inner core and the mantle do not change either, as (4.13-14) show.

The viscous controlled approximation

Turbulence effectively enhances the viscosity and respectively enhances λ as well. As (2.3) shows, the maximum possible values of λ at $\nu=\eta$ is of order of $3\text{x}10^1$. Therefore this viscous controlled case, λ » 1, is also a problem of physical interest. It is additionally an interesting one, because all the computer simulations implicitly suppose that λ » 1. For example, Glatzmaier and Roberts (1995,1996) in fact carried out their simulations for $E \sim 10^{-4}$ and $R_o \sim 10^{-5}$. To compare our estimation with the results of the computer simulations, we consider in this section the case of large viscosity ($\lambda \to \infty$).

If we adopt λ » 1 then it follows from (6.3) that the flux in the bulk of the core depends on *s* only: $\nabla \cdot \mathbf{1}_z \lambda \overline{\chi}^{(1)} = 0 \to \overline{\chi}^{(1)} = \overline{\chi}^{(1)}(s)$. It follows from here that the next approximation meridional flow has only a *z* component:

$$\overline{\mathbf{V}}^{(1)} = -\frac{\mathbf{1}_\phi}{\overline{\rho}(r)s} x\nabla\overline{\chi}^{(1)}(s) = \frac{\mathbf{1}}{\overline{\rho}(r)s}\frac{\partial\overline{\chi}^{(1)}(s)}{\partial s} = \mathbf{1}_z \overline{V}_z^{(1)}(s, \sqrt{s^2+z^2}), \tag{9.1}$$

The space density of the mass flux $\overline{\rho}(r)\overline{V}^{(1)} = (1_z / s)(\partial\overline{\chi}^{(1)}(s)/\partial s)$ of this flow does not depend on *z*. Respectively the flux coming out from the Ekman layer on the bottom boundary of the liquid core enters without any changes into the layer on the top boundary at any *s*. The averaged over ϕ boundary condition of the next approximation (2.9,10) can be rewritten in the form:

$$\frac{\partial\overline{\chi}_\pm^{(1)}(s)}{sds} = \pm 2\overline{\gamma}(\mathbf{r}_{\pm 1})\omega_1 - \frac{\overline{\gamma(\mathbf{r}_{\pm 1})\mathbf{r}_{\pm 1}\cdot\mathbf{W}^{(0)}(\mathbf{r}_{\pm 1})}}{z_1}, \quad (=\overline{\rho}(r_1)\overline{V}_z^{(1)}(s, \pm z_1)) \tag{9.2}$$

$$\frac{\partial\overline{\chi}_\pm^{(1)}(s)}{sds} = \pm 2\overline{\gamma}(\mathbf{r}_{\pm 2})\omega_1 - \frac{\overline{\gamma(\mathbf{r}_{\pm 2})\mathbf{r}_{\pm 2}\cdot\mathbf{W}^{(0)}(\mathbf{r}_{\pm 2})}}{z_2}, \quad (=\overline{\rho}(r_2)\overline{V}_z^{(1)}(s, \pm z_2)), \tag{9.3}$$

where $\mathbf{W} = \nabla \times \mathbf{V}$.

Outside the tangent cylinder, ($s > r_1$), the bottom and the top boundaries are $-z_2(s)$ and $z_2(s)$. Then by equalizing rhs of (9.3) at $-z_2(s)$ and $z_2(s)$ we obtain:

$$\frac{\overline{\gamma(\mathbf{r}_2)\mathbf{r}_2\cdot\mathbf{W}^{(0)}(\mathbf{r}_2)} - \overline{\gamma(\mathbf{r}_{-2})\mathbf{r}_{-2}\cdot\mathbf{W}^{(0)}(\mathbf{r}_{-2})}}{z_2} = 2\left[\overline{\gamma}(\mathbf{r}_2) + \overline{\gamma}(\mathbf{r}_{-2})\right]\omega_2; \quad r_1 < s < r_2. \tag{9.4}$$

Inside the tangent cylinder the bottom and the top boundaries are $z_1(s)$ and $z_2(s)$ in the upper hemisphere and $-z_2(s)$ and $-z_1(s)$ in the lower one. Then combining (9.2) and (9.3) we obtain the equations for $\mathbf{W}^{(0)}$ inside the tangent cylinder in the North and the South hemispheres:

$$\frac{\overline{\gamma(\mathbf{r}_{\pm 2})\mathbf{r}_{\pm 2}\cdot\mathbf{W}^{(0)}(\mathbf{r}_{\pm 2})}}{z_2} + \frac{\overline{\gamma(\mathbf{r}_{\pm 1})\mathbf{r}_{\pm 1}\cdot\mathbf{W}^{(0)}(\mathbf{r}_{\pm 1})}}{z_1} = \pm 2\left[\overline{\gamma}(\mathbf{r}_{\pm 1})\omega_1 + \overline{\gamma}(\mathbf{r}_{\pm 2})\omega_2\right], \quad 0 < s < r_1 \tag{9.5}$$

Let us remind that the whole flow of the leading approximation has a form: $V^{(0)} = \mathbf{1}_\phi \overline{V}^g(s) + \mathbf{1}_\phi \overline{V}^F(s,z) - (\mathbf{1}_\phi / \overline{\rho}(r)s) \times \nabla\overline{\chi}^{(0)}(s,z) + \tilde{\mathbf{V}}^{(0)}(s,\phi,z)$. The only unknown quantity in this flow is the geostrophic velocity $\overline{V}^g = s\overline{\omega}^g(s)$. The normal component of the curl takes the form: $\mathbf{r}.\overline{\mathbf{W}}^{(0)} = \mathbf{r}.\overline{\mathbf{W}}^g + \mathbf{r}.\overline{\mathbf{W}}^F + \mathbf{r}.\tilde{\mathbf{W}}^{(0)}$ where

$$\mathbf{r}.\overline{\mathbf{W}}^{g} = \pm\frac{|z|}{s}\frac{\partial s^2\overline{\omega}^{g}_{\pm}}{\partial s};\qquad \mathbf{r}.\overline{\mathbf{W}}^{F} = \pm\frac{|z|}{s}\frac{\partial s^2\overline{\omega}^{F}}{\partial s}+s\mathbf{1}_{\phi}.\nabla\times\frac{\overline{\mathbf{F}}}{\overline{\rho}}$$

in the North and the South hemispheres respectively.

Thus (9.4), (9.5) are in fact equations for the angular geostrophic velocity $\overline{\omega}^{g}(s)$. Taking (9.4) into account we obtain equations for this value outside the tangent cylinder

$$\frac{\partial s^2\left[\overline{\omega}^{g}(s)-\omega_2\right]}{sds} = \frac{\overline{\gamma}(\mathbf{r}_{-2})\mathbf{r}_{-2}\cdot\overline{\mathbf{W}}^{F}(\mathbf{r}_{-2})-\overline{\gamma}(\mathbf{r}_2)\mathbf{r}_2\cdot\overline{\mathbf{W}}^{F}(\mathbf{r}_2)}{z_2(\overline{\gamma}(\mathbf{r}_{-2})+\overline{\gamma}(\mathbf{r}_2))} - \frac{\overline{\tilde{\gamma}(\mathbf{r}_2)\mathbf{r}_2\cdot\tilde{\mathbf{W}}^{(0)}(\mathbf{r}_2)}-\overline{\tilde{\gamma}(\mathbf{r}_{-2})r_{-2}\cdot\tilde{\mathbf{W}}^{(0)}(\mathbf{r}_{-2})}}{z_2(\overline{\gamma}(\mathbf{r}_{-2})+\overline{\gamma}(\mathbf{r}_2))}\qquad r_1 < s < r_2 \tag{9.6}$$

and inside it in the North and the South hemispheres

$$\frac{\partial s^2\left[\overline{\omega}^{g}_{\pm}(s)-\omega_2\right]}{sds} = 2\frac{\overline{\gamma}(\mathbf{r}_{\pm1})\left[\omega_1-\omega_2\right]}{\overline{\gamma}(\mathbf{r}_{\pm1})+\overline{\gamma}(\mathbf{r}_{\pm2})} \mp \frac{\overline{\gamma}(\mathbf{r}_{\pm1})r_{\pm1}\cdot\overline{\mathbf{W}}^{F}(\mathbf{r}_{\pm1})}{(\overline{\gamma}(\mathbf{r}_{\pm1})+\overline{\gamma}(\mathbf{r}_{\pm2}))z_1} \mp \frac{\overline{\gamma}(\mathbf{r}_{\pm2})\mathbf{r}_{\pm2}\cdot\overline{\mathbf{W}}^{F}(\mathbf{r}_{\pm2})}{(\overline{\gamma}(\mathbf{r}_{\pm1})+\overline{\gamma}(\mathbf{r}_{\pm2}))z_2} \mp \frac{\overline{\tilde{\gamma}(\mathbf{r}_{\pm1})\mathbf{r}_{\pm1}\cdot\tilde{\mathbf{W}}^{(0)}(\mathbf{r}_{\pm1})}}{(\overline{\gamma}(\mathbf{r}_{\pm1})+\overline{\gamma}(\mathbf{r}_{\pm2}))z_1} \mp \frac{\overline{\tilde{\gamma}(\mathbf{r}_{\pm2})\mathbf{r}_{\pm2}\cdot\tilde{\mathbf{W}}^{(0)}(\mathbf{r}_{\pm2})}}{(\overline{\gamma}(\mathbf{r}_{\pm1})+\overline{\gamma}(\mathbf{r}_{\pm2}))z_2}\qquad r_1 < s < r_2 \tag{9.7}$$

Equations (9.6), (9.7) allow to find $\overline{\omega}^{g}$ inside and outside the tangent cylinder and so to define the whole flow of the leading approximation.

In the last part of this section we will discuss the azimuthal flow near the axis OZ. Any continuous function $\Phi(s,\phi,z)$ near this axis does not depend on ϕ. Respectively, $\overline{\gamma}$ vanisheshere and so $\overline{\gamma\mathbf{r}.\mathbf{W}^{(0)}}$ takes a simple form: $\overline{\gamma\mathbf{r}.\mathbf{W}^{(0)}} = \overline{\gamma}\mathbf{r}.\overline{\mathbf{W}}^{(0)}$ where

$$\mathbf{r}.\overline{\mathbf{W}}^{(0)} = \frac{z}{s}\frac{\partial s\overline{V}^{(0)}_{\phi}}{\partial s} - s\frac{\partial\overline{V}^{(0)}_{\phi}}{\partial z} = \frac{z}{s}\frac{\partial s^2\overline{\omega}^{(0)}}{\partial s} - s^2\frac{\partial\overline{\omega}^{(0)}}{\partial z} \sim 2z\overline{\omega}^{(0)}(z);\qquad \left(\overline{\omega}^{(0)} = \frac{\overline{V}^{0}_{\phi}}{s}\right)$$

As a result, (9.5) in the North and the South hemispheres near this axis can be written as:

$$\overline{\gamma}(\pm z_2)\overline{\omega}^{(0)}(\pm z_2) = -\overline{\gamma}(\pm z_1)\overline{\omega}^{(0)}(\pm z_1)+\overline{\gamma}(\pm z_1)\omega_1+\overline{\gamma}(\pm z_2)\omega_2\quad \text{at } s = 0. \tag{9.9}$$

Our scale for the magnetic field $\sqrt{2\Omega\eta\overline{\rho}_1\mu_o} = 1.9\,\text{mT}$ is a rather big one. So one can assume that the dimensionless magnetic field is small: $B_r^2 << 1$ at the boundaries. Then it follows from (2.5) that $\overline{\gamma}(r_{1,2}) \approx \overline{\rho}(r_{1,2})/\sqrt{2}$. We neglect the small difference between $\overline{\rho}_1$ and $\overline{\rho}_2$ and ω_2 in comparison with ω_1 and reduce (9.9) to the form:

$$\overline{\omega}^{(0)}(\pm z_2) \approx -\overline{\omega}^{(0)}(\pm z_1)+\omega_1 \qquad \text{at } s = 0 \tag{9.10}$$

It follows from (5.4) that

$$\overline{\omega}^{(0)}(\pm z_2)-\overline{\omega}^{(0)}(\pm z_1)=\Delta\omega_{\pm}^{F}, \quad \Delta\omega_{\pm}^{F}=-\int_{z_1}^{z_2} dz\nabla.\left[\frac{\mathbf{1}_\phi}{s}\times\frac{\overline{\mathbf{F}}(0,\pm z)}{\overline{\rho}}\right] \qquad \text{at } s=0 \qquad (9.11)$$

By combining (9.10) and (9.11) we can the estimate the values of the local angular velocity $\overline{\omega}(s,z)$ on the outer sides of the layers on CMB and ICB near the poles of the liquid and the solid core:

$$\overline{\omega}^{(0)}(\pm z_2)\approx\frac{\Delta\omega_{\pm}^{F}+\omega_1}{2}\ ;\ \overline{\omega}^{(0)}(\pm z_1)\approx\frac{\Delta\omega_{\pm}^{F}+\omega_1}{2} \qquad \text{at } s=0\ . \qquad (9.12)$$

These expressions show that in the case when $\left|\Delta\omega_{\pm}^{F}\right| \gg \omega_1$ values $\overline{\omega}^{(0)}(z_2)$ and $\overline{\omega}^{(0)}(z_1)$ are approximately of the same absolute value, but with opposite signs. Contours of the axisymmetric azimuthal angular velocity in the computer simulations (Glatzmaier and Roberts, Fearn and Morrison and others) show that this conclusion is in a good agreement with the computer results.

From (5.1,2) we can obtain not only the difference between values $\overline{\omega}^{(0)}$ on the poles of the inner and outer core, but the *z* dependence of $\overline{\omega}^{(0)}(z)$ on the axis OZ as well:

$$\overline{\omega}^{(0)}(z)=\overline{\omega}^{(0)}(z_1)+\frac{\overline{F}_s(s,z_1)-\overline{F}_s(s,z)}{s\overline{\rho}}-\int_{z_1}^{z} dz\frac{1}{s}\frac{\partial}{\partial s}\frac{\overline{F}_z}{\overline{\rho}} \qquad \text{at } s=0. \quad (9.13)$$

The force in this formula is a sum of the Archimedean and the Lorentz forces. Respectively, the azimuthal flow strength $\overline{\omega}^{(0)}(z)$ can be presented as a sum of the components $\overline{\omega}^{a(0)}(z)$ and $\overline{\omega}^{m(0)}(z)$. Taking into account the buoyancy force $\overline{\mathbf{F}}^{a}=\mathbf{r}A(r)\overline{\rho}\overline{C}(s,z)$ one can obtain the Archimedean part of $\overline{\omega}^{(0)}(z)$ at the axis OZ:

$$\overline{\omega}^{a(0)}(z)=\overline{\omega}^{a(0)}(z_1)+\text{const}\left[\overline{C}(z_1)-\overline{C}(z)-\int_{z_1}^{z} dz\frac{z}{s}\frac{\partial}{\partial s}\overline{C}(s,z)\right] \quad \text{at } s=0\ . \ (9.13a)$$

If we assume that the temperature at the axis OZ depends only on *r*, then we obtain: $(z/s)(\partial\overline{C}(r)/\partial s)=(z/r)(\partial\overline{C}(r)\partial r)=\partial\overline{C}(r)/\partial z$. In this case $\overline{\omega}^{a(0)}(z)$ converts into a linear function of temperature:

$$\overline{\omega}^{a(0)}(z)=\overline{\omega}^{a(0)}(z_1)+\text{const}[\overline{C}(z_1)-\overline{C}(z)] \qquad \text{at } s=0. \qquad (9.13b)$$

Since the temperature is a monotonous function of *z*, $\overline{\omega}^{a(0)}(z)$ has to be monotonous as well.

Discussion

The momentum equation for the flow in the Earth's core (1.1,2) is commonly written in a form which depends on two small parameters, the Ekman, E, and the Rossby, R_o, numbers. We rewrite this equation in a form (2.1,2) which depends on other two parameters $\sqrt{E}$, and $\lambda = \sqrt{E}/R_o$. The first of them is small yet but the other one is not small and exceeds the first one with approximately six orders. Boundary conditions (2.4) depend only on the first parameter $\sqrt{E}$. Therefore we search for the solution of the problem in the form of decomposition (2.6) in this parameter. Then the flow of the leading approximation can be presented as a sum of the geostrophic and the force flows:

$$\mathbf{V}^{(0)} = \mathbf{1}_\phi \overline{V}^g(s,t) + \mathbf{V}^F$$

$$\mathbf{V}^F(s,\phi,z) = \mathbf{1}_\phi \overline{V}_\phi^F(s,z) - \frac{\mathbf{1}_\phi}{\overline{\rho}(r)s} \times \nabla \overline{\chi}^{(0)}(s,z) + \tilde{\mathbf{V}}^F(s,\phi,z) \tag{10.1}$$

The force flow is defined directly by the force and satisfies the no-normal boundary conditions (2.7,8). The non-axisymmetric meridional flow has two, $\tilde{V}_s^F(s,\phi,z)$ and $\tilde{V}_z^F(s,\phi,z)$, components which can be defined by these conditions on the top and the bottom boundaries at any cylinder with radius *s*. Then the ϕ component of this flow, $\tilde{V}_\phi^F(s,\phi,z)$, can be obtained from the continuity equation (2.2). The axisymmetric meridional flow depends on only one function $\overline{\chi}^{(0)}(s,z)$, but it also has to satisfy this condition on the top and the bottom boundaries. These two conditions for the only function $\overline{\chi}^{(0)}(s,z)$ are over-constraint for the problem. That is why an additional condition (4.3) has to be satisfied in the liquid core. This condition is the *Taylor constraint*.

The force flow establishes instantly by means of pressure. Respectively, its time dependence is defined wholly by the time dependence of the force. Thus it does not have its own time dependence. To emphasize this we do not include *t* in the list of its variables.

The geostrophic flow is defined by two mechanisms: (6.8,9) concerned to fluxes of the angular momentum and (6.11-14) related to the Ekman suction into the boundary layers. Let us discuss primarily the first of them.

As (6.3) shows, the geostrophic flow is defined by two fluxes of the angular momentum. The first one, $1_z \lambda \overline{\chi}$, describes transport in *s* direction of the angular momentum connected to the angular velocity $1_z \Omega$ of the reference framework. Moving in *s* direction, a liquid parcel conserves the ϕ component of its velocity related to the rotation of the framework. Respectively the angular velocity of the parcel changes and this change is proportional to the mass flux $\Delta\chi$ associated with the parcel. If we take the cylinder shell between the top and the bottom boundaries qua this parcel, then $\Delta\chi$ vanishes in the leading approximation as (4.3) shows. That is why the angular velocity of the geostrophic flow is defined by the difference of fluxes of the next approximation (see e.g. the second term in lhs of (6.8)). Note that the term $s^2 R(s)$ plays the role of the inertia moment of the cylinder shell.

The other flux $\overline{\rho} s \overline{\mathbf{V}_m V_\phi}$ of the angular momentum in (6.3) is related to the transfer of the angular momentum density $\overline{\rho} s V_\phi$ by the meridional flow $\mathbf{V}_m$ averaged over ϕ. This flux creates lhs of (6.8,9) and (6.17).

The second (6.11-14) mechanism defining the geostrophic flow is related to the Ekman suction into the boundary layers. Both the geostrophic and the force flows create the suction, but only the first one can fit together with the boundary mass flux, since the second one is already defined by the force. This flux is restricted by the mass conservation. So it establishes instantly (only in a few minutes) by means of pressure. That is why these equations do not depend on time, opposite to the "momentum" equations (6.8,9).

Defining the force flow from the equations (5.1,2) (5.3) and (4.2) we obtain the functions which are included in equations (6.8,9) and (6.11-14) for the geostrophic flow. Then the solution of these equation gives the whole flow.

The *Taylor state* is called an assumed solution which does not depend on the Ekman number in the inviscid limit $E \to 0$. The force flow obeying the no-normal-flow boundary condition does not depend on *E*. *So only the geostrophic flow, which depends on the Ekman number E by means of* λ *in (6.8,9) and (6.17), disturbs the Taylor state conditions.* There are two limits, the inviscid one $\lambda \to 0$ and the "viscous controlled" limit $\lambda \to \infty$ when this disturbance is violated and the whole flow converts into Taylor state.

The inviscid flow is of a real geophysical interest since $\lambda = \sqrt{2\Omega\nu} / V_1$ for the kinematic values of viscosity is a small one, $\lambda \sim 10^{-2}$. In this discussion we fix the velocity scale, V_1, by the typical value of the westward drift. If we admit that the velocity scale can change, then we could call it a case of fast flow (or a case of slow rotation, supposing that Ω can change as well, which could be interesting in respect of Venus).

We consider the inviscid limit in section 8. The geostrophic flow in this limit is defined by the equations (8.1-3) in which rhs is defined by the force flow of the leading approximation. So the whole flow is determined by the no-normal boundary conditions and thus it does not depend on the Ekman suction and respectively on *E*. This is in agreement with the heuristic expectation that viscosity which is small enough cannot influence the flow. The equations (3.5) and (4.11-14) show that the kinetic energy, the angular momentum of the liquid core and the angular velocities of the inner core and the mantle do not change in this approximation.

It is important to emphasize that the opposite (the viscous controlled) limit is also applicable in context of the Geodynamo, since as (2,3) shows, λ is relatively great for the turbulent viscosity. This means that the results of the computer simulations which assume implicitly λ » 1 have a physical meaning. This is not trivial, since if we adopt the same values of η and V_1 for Venus (where Ω is smaller with two orders of magnitude) then λ becomes of order of 1 and the viscous controlled numerical results could be unapplicable to this planet. But, as mentioned above, these results are applicable for the Earth. Moreover, being independent of λ (and respectively of *E*) the computer simulated flows convert into Taylor state and thus describe adequately the flow in the Earth's core. This is also a non-trivial conclusion, taking into account that the typical values of the Ekman numbers in computer simulations exceed these in the Earth core, typically with ten orders of magnitude.

In the viscous controlled case the geostrophic flow loses its own time dependance and is defined inside and outside the tangent cylinder by the Ekman suction on the boundaries (9.6,7). How does this reflect on the flow? Let us consider e.g. the energy equation (3.5). The time derivative of the kinetic energy, as (3.5) shows, is equal to the difference between the force work and the layer dissipation, multiplied by λ. In the case of large λ this difference tends to be zero. So in the computer simulations it is defined mainly by random computer errors. A large parameter λ enhances essentially ($\sim 10^3$) their influence. Therefore one can expect that the time derivative of the kinetic work must be very discontinuous. Fig 1 of Glatzmaier and Roberts (1995) confirms this conclusion.

As we noticed above, the computer simulated flows are at asymptotically large values of λ. For example, Glatzmaier and Roberts (1995,1996) in fact carried out their simulations for $E \sim 10^4$ and $R_o \sim 10^{-5}$, i.e. for $\lambda \sim 10^3$. In order to decrease this large value to the maximal real one ($\lambda \sim 30$), these authors have to diminish E with three orders of magnitude, which is hardly possible for the existing computers. All the more it is impossible to obtain a computer solution with a small λ, implying the kinematic viscosity in E. That is why we propose to use another way which is based on decomposition of the flow (2.6). This implies the creation of a computer code for solving the equations presented here. This code would be free of the difficulties concerning the resolution of thin boundary layers.

The axisymmetric part of the meridional force flow can be obtained by the integration of (4.2) under boundary conditions (2.13). Its azimuthal part is defined by the equation (5.4). The non-axisymmetric part of the force flow can be analytically obtained from the equations (5.1,2) and the boundary conditions (2.7,8). This problem also can be solved numerically.

Having evaluated the force flow, we can search for the only component of the flow which is not defined by the leading approximation problem, the geostrophic one. Equations (6.8-14) define it inside the tangent cylinder and (6.17,19) do this outside the tangent cylinder.

The problem is simplified additionally in the inviscid limit $\lambda \to 0$. Then we obtain the only simple equations (8.1-3) for the geostrophic velocity, which can be solved numerically.

The equations (9.6,7) for the viscous controlled case of large λ can be used for comparing our results with the numerical solutions. Let us emphasize here that not only the whole geodynamo simulations, but also some simple models such as "2.5" can be used for the test.

References

Glatzmaier, G.A., and P.H. Roberts, 1995. A three-dimensional convective dynamo solution with rotating and finitely conducting inner core and mantle, *Phys. Earth Planet. Inter.*, **91**, 63-75.

Glatzmaier, G.A., and P.H. Roberts, 1996a. An anelastic evolutionary geodynamo simulation driven by compositional and thermal convection, *Physica*, *D***97**, 81-94.

Glatzmaier, G.A., and P.H. Roberts, 1996b. Rotation and magnetism of Earth's inner core, *Science*, **274**, 1887-1891.

Roberts, P.H., and G.A. Glatzmaier, 2000. Geodynamo theory and simulations, *Reviews of Modern Physics*, **72**, 1081-1123.

Jones C.A., A.W. Longbotton, R. Hollerbach, 1995 A self-consistent convection driven geodynamo model, *Phys. Earth Planet. Inter*., **92**, 119-141.

Fearn, D.R. and G.Morrison, 2001. The role of inertia in hydrodynamic models of the geodynamo, *Phys. Earth Planet. Inter.*, **128**, 75-92

Върху задачата за течението на течното ядро на земята

А.Ануфриев, М.Тасев

Резюме: Хидродинамичното уравнение за течното ядро на Земята, зависещо от два малки параметъра (числата на Ekman - E , и Rossby -), е преобразувано към форма, която зависи от други два параметъра. ($\sqrt{E}$ и $\lambda = \sqrt{E}/R_o$) . Първият от тях е все още малък, но вторият надвишава $\sqrt{E}$ с близо шест порядъка. Граничните условия на течението зависят само от $\sqrt{E}$. Тъй като и уравнението и граничните условия зависят от един и същ малък параметър $\sqrt{E}$, ние търсим решението на хидродинамичната задача във вид на развитие по този параметър. Получените уравнения показват, че течението в главното приближение в развитието по $\sqrt{E}$ зависи само от параметъра λ . Това означава, че за бързо въртеливите течности $\lambda = \sqrt{2\Omega\nu}/V_1$ става универсален параметър (каквото е например числото на Рейнолдс $R = V_1 L_1/\nu$ в хидродинамиката), и всички теченията могат да се класифицират само чрез него. Пълното течение може да бъде представено като сума от две компоненти – силово и геострофичното течения. Първото може да бъде определено пряко чрез интегралите от силата. Второто се подчинява на уравненията, зависещи от производната по време на геострофичната скорост. По този начин геострофичното течение е единствената част от цялото течение, която има собствено времево поведение. То еволюира заедно с магнитното поле и с температурните разпределения. Противоположно на това, силовата част на течението се съгласува с моментното силово разпределение, и нейното времево поведение се определя от това на силата. Могат да бъдат разгледани два възможни предела: невизкозен ($\lambda \to 0$) и вискозно контролируем ($\lambda \to \infty$), в които течението не зависи от λ ($= \sqrt{E}/R_o$) и, следователно, от E. Затова течението се преобразува в Тейлърово състояние в тези предели. В зависимост от вискозитета, параметърът λ за Земното ядро се изменя от 2.3×10^{-2} за кинематичния вискозитет до 3.3×10^{1} за най-голямата стойност на турбулентния вискозитет $\nu_T = \eta$, където η е коефициент на магнитната дифузия. Така и двата типа течения от типа на Тейлъровото състояние могат да се реализират в Земното ядро. От гледна точка на компютърното моделиране вискозно контролируем

случай е особено важен, тъй като всички компютърни симулации използват неявно предположение, че λ » 1. Например, Глатцмайер и Робертс (1995, 1996) фактически решават задача за случая $\lambda \sim 10^3$. По такъв начин тяхното асимптотично (при $\lambda \to \infty$) решение е приблизително подходящо за условията на Земното ядро с („*турбулентно*”) $\lambda = 3.3 \times 10^1$. Това не е тривиално, тъй като при същите стойности на $\nu\_(= \nu_T = \eta)$ и V_1 за Венера, където Ω е с два порядъка по-малко от земното, λ става от порядъка на 1 и “вискозно контролирани" числени резултати, стават неприложими към тази планета. Нещо повече. Бъдейки независимо от λ (и съответно от Е), визкозно контролируемите течения представляват Тейлърово състояние. И затова при турбулентен визкозитет Тейлъровото състояние адекватно описва течението в Земното ядро. Това също е нетривиално заключение, ако се вземе под внимание, че характерните стойности на Екмановото число в компютърните симулации обикновено надвишават тези в Земното ядро с десет порядъка! Макар и компютърните модели да описват задоволително течението с голямо λ , те са неприложими към за решаване на задачата с по-малки стойности на λ - и особено в невизкозитения граничен случай $\lambda \to 0$, поради необходимостта от числено разрешаване (резолюция) на тънките гранични слоеве. Смятаме, че тази трудност ще може да бъде преодоляна със създаването на машинни кодове на основата на уравненията, представени тук, при които проблемът с резолюцията отсъства .