ADIABATIC APPROXIMATION IN THE GEODYNAMO CONVECTION

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Abstract. Rotating stars and planets possess cylindrical symmetry and therefore it is natural to decompose their magnetic field into azimuthal and meridional components: $\mathrm{B}_a = 1_\phi B_\phi$ and , $\mathrm{B}_m = \mathrm{B} - \mathrm{B}_a$ respectively. We derived the induction equations for axisymmetric azimuthal and meridional fields, as well as for non-axisymmetric ones with zero mean values. As is normally accepted, we express the axisymmetric meridional field by the stream line function: $\overline{\mathrm{B}}_m = 1_\phi \times s^{-1} \nabla \psi(s, z)$ Our equations are derived for compressible liquid. Nevertheless, in absence of electromotive force (emf) they coincide in form with the equations for the axisymmetric field for incompressible liquid (see e.g. Moffatt (1978)). However, the compressibility of the liquid plays an essential role in these equations. This leads to the appearance of new generating terms in the equations for the poloidal and the toroidal fields. These terms allow the generation in a pure axisymmetric case. Nevertheless, this does not disturb Cowling theorem which forbids steady axisymmetric generation, but not axisymmetric generation at all. Based on our equations we analyze the magnetic field generation in the Earth core. This analysis shows that powerful generation of the field is possible in the boundary layers near ICB and CMB. This generation is due to the α and ω effects occurring under the condition that the parameter $R_m \hat{\delta}$ is not small, $\hat{\delta}$ and R_m being the thickness of the layer and magnetic Reynolds numbers, respectively. In such a case the main part of the mean magnetic field upper and under inner core is supported by this layer generation. As such, the effect of the layer generation on the magnetic field depends on the value of the above parameter. If $R_m \hat{\delta} \geq 1$, the layer generation is dominant; in the opposite case, $R_m \hat{\delta} << 1$ it is weak and results only in small perturbations on the boundaries. Since $\hat{\delta}$ is proportional to $\sqrt{E}$ we conclude that in the Taylor state (solution at $E \to 0$) the layer generation is not essential. However, the layer generation can be very important in numerical simulations since E are typically not small enough. This question is discussed in sections 3 and 4.

Key words: dynamo theory, boundary layer, α and ω effects.

Induction equation

Spherical symmetry of the generating regions of stars and planets makes suitable commonly used decomposition of theirs magnetic fields, $\mathbf{B}$, into the toroidal, $\mathrm{B}_T = \nabla\times \mathrm{r}T(\mathrm{r}) = -\mathrm{r}\times\nabla T$, and the poloidal, $\mathrm{B}_p = \nabla\times\nabla\times \mathrm{r}P(\mathrm{r}) = -\nabla\times\mathrm{r}\times\nabla P$, parts: $\mathbf{B}=\mathbf{B}_T+\mathbf{B}_P$.
However rotation of stars and planets disturbs the spherical symmetry reducing it to the axial one. That is why the axial decomposition of the vectors fields seems to be more relevant for this case. It implies presentation of the field, $\mathbf{B}$, as a sum of the azimuthal, $\mathbf{B}_a=\mathbf{1}_\phi B_\phi$, and the meridional, $\mathbf{B}_m=\mathbf{B}-\mathbf{B}_a$, components, where $\mathbf{1}_\phi$ is the unit vector in the ϕ direction of the cylindrical polar, (s, ϕ , z) or spherical (r,ϑ,ϕ) coordinates. Though most of the speculations in this paper are carried out for the cylindrical polar coordinates their results are valid for the spherical polar coordinates (r,θ,ϕ) as well.
Notice that in the case of axisymmetric fields their meridional parts coincide with the poloidal and azimuthal with the toroidal fields. That is why for this case we will use their more customary names.
Our goal in this work is to derive the induction equations for the axisymmetric azimuthal and the meridional fields and for their non-axisymmetric counterparts. To make this we substitute into the Ohm's and the Maxwell equations

$$\mathrm{j} = \sigma[\mathrm{E}+\mathrm{V}\times\mathrm{B}], \qquad \nabla\times\mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}, \qquad \mathrm{j} = \frac{\nabla\times\mathrm{B}}{\mu_o}, \tag{1.1,2,3}$$

the axial decomposition of the magnetic field:

$$\mathrm{B} = \mathrm{B}_a + \mathrm{B}_m, \qquad \mathrm{B}_a = \frac{1_\phi}{s} sB\phi, \qquad \mathrm{B}_m = \frac{1_\phi}{s}\times\mathrm{A}_m, \qquad \mathrm{A}_m = -1_\phi\times s\mathrm{B}. \tag{1.4}$$

As here we will often use vector $\mathbf{1}_\phi/s$ instead of $\mathbf{1}_\phi$. It is more convenient for handling, since the divergence and the curl of it equals zero.

Maxwell equation (1.2) can be expressed (see (A.3) in Appendix) in terms of azimuthal and meridional components:

$$\frac{\partial \mathrm{B}_a}{\partial t} = 1_\phi s\nabla\cdot\left[\frac{1_\phi}{s}\times\mathrm{E}_m\right], \qquad \frac{\partial \mathrm{B}_m}{\partial t} = -\left[\frac{1_\phi}{s}\times\frac{\partial \mathrm{E}_m}{\partial\phi}\right] + \frac{1_\phi}{s}\times\nabla sE_\phi. \tag{1.5,6}$$

Expressing $\mathbf{E}$ from (1.1) we obtain:

$$\mathrm{E}_m = \frac{\mathrm{j}_m}{\sigma} - [\mathrm{V}\times\mathrm{B}]_m = \frac{\mathrm{j}_m}{\sigma} - 1_\phi \times s\mathrm{F}_m, \qquad E_\phi = \frac{j_\phi}{\sigma} - [\mathrm{V}\times\mathrm{B}]_\phi = \frac{j_\phi}{\sigma} - F_\phi, \tag{1.7,8}$$

where (see (A4), (A5))

$$s\mathrm{F}_m = -1_\phi \times [\mathrm{V}\times\mathrm{B}] = \mathrm{B}_m V_\phi - \mathrm{V}_m B_\phi,$$
$$F_\phi = 1_\phi \cdot [\mathrm{V}_m \times \mathrm{B}_m] = [V_z B_s - V_s B_z] = [V_r B_\vartheta - V_\vartheta B_r]$$

Substitution (1.7,8) into (1.5,6) gives the induction equation in the form:

$$\frac{\partial}{\partial t}\frac{B_\phi}{s} = \nabla\cdot\mathrm{F}_m + \nabla\cdot\frac{\eta}{s^2}\nabla s^2\frac{B_\phi}{s} - \frac{\partial}{\partial\phi}\left(\mathrm{B}\cdot\nabla\frac{\eta}{s^2}\right), \tag{1.9}$$

$$\frac{\partial\mathrm{B}_m}{\partial t} = -\frac{\partial\mathrm{F}_m}{\partial\phi} + \frac{1_\phi}{s}\times\nabla sF_\phi - [\nabla\times[\eta\nabla\times\mathrm{B}]]_m, \tag{1.10}$$

where $\eta = 1\sigma\mu_o$ and, bearing in mind (A6),

$$[\nabla\times[\eta\nabla\times\mathrm{B}]]_m = \frac{\eta}{s^2}\frac{\partial}{\partial\phi}\left[\frac{\partial\mathrm{B}_m}{\partial\phi} - \nabla_m sB_\phi\right] - \frac{1_\phi}{s}\times\nabla\left[\eta s^2\nabla\cdot\left[\frac{1_\phi}{s}\times\mathrm{B}_m\right]\right]. \tag{1.11}$$

When a well mixed fluid is assumed, η is uniform in the whole liquid core. Then the expression for $[\nabla\times[\eta\nabla\times\mathrm{B}]]_m$ can be simplified (see (A8)):

$$[\nabla\times[\eta\nabla\times\mathbf{B}]]_m = \eta[\nabla^2\mathbf{B}]_m = \mathbf{1}_s\frac{\eta}{\mathbf{s}}\left[\nabla\cdot s^2\nabla\frac{B_s}{s} - \frac{2}{s}\frac{\partial B_\phi}{\partial\phi}\right] + \mathbf{1}_z\eta\Delta B_z. \tag{1.12}$$

At the boundaries with solid bodies the mantle and inner core η can change sharply. That is why only derivatives in respect of r in (1.11) and (1.9) have to be taken into consideration in these regions:

$$\nabla\times[\eta\nabla\times\mathbf{B}] = -\mathbf{1}_r\eta\frac{\partial^2 B_r}{\partial r^2} - \mathbf{1}_\vartheta\frac{\partial}{\partial r}\eta\frac{\partial B_\vartheta}{\partial r} - \mathbf{1}_\phi\frac{\partial}{\partial r}\eta\frac{\partial B_\phi}{\partial r} \tag{1.13}$$

$$\mathbf{F}_m = \mathbf{B}_m (V_\phi / s) - \mathbf{V}_m (B_\phi / s)$$

$$\frac{\partial}{\partial t}\frac{B_\phi}{s} = -\omega\frac{\partial}{\partial \phi}\frac{B_\phi}{s} + \nabla\cdot\frac{\eta}{s^2}\nabla s^2\frac{B_\phi}{s} - \frac{\partial}{\partial \phi}\left(\mathbf{B}\cdot\nabla\frac{\eta}{s^2}\right)$$

$$\frac{\partial \mathbf{B}_m}{\partial t} = -\omega\frac{\partial \mathbf{B}_m}{\partial \phi} - [\nabla\times[\eta\nabla\times\mathbf{B}]]_m$$

$\nabla\cdot\mathrm{B}=0$ $\nabla\cdot\mathrm{j}=0$

$\{B_n\}=0, \{j_n\}=0$

$\{\mathrm{E}\}_\tau = 0, \{\mathrm{B}\}_\tau = 0$

$\{\mathrm{j}/\sigma\}_\tau = \{\mathrm{E}_\tau\} + \{\mathrm{V}\times\mathrm{B}\}_\tau = 0$

$\nabla\cdot\mathrm{B}=0$ $\{\mathrm{B}\}_\tau = 0$

$$\{\mathrm{B}\}=0, \quad \left\{\eta\frac{\partial B_{\vartheta,\phi}}{\partial r}\right\}=0, \quad \left\{\frac{\partial B_r}{\partial r}\right\}=0.$$

$\nabla\times[\eta\nabla\times\mathrm{B}]$

$\mathrm{V}=\tilde{\mathrm{V}}+\overline{\mathrm{V}}$

$\mathbf{B}=\tilde{\mathbf{B}}+\overline{\mathbf{B}}$ $\overline{\mathbf{V}}$ $\overline{\mathbf{B}}$

$\overline{\mathbf{V}}$ $\overline{\mathbf{B}}$

$\nabla\cdot\mathrm{B}=0$

$\nabla\cdot\overline{\mathbf{B}}_m(s,z)=0$

$$\nabla\cdot\overline{\mathbf{B}}_m = \nabla\cdot[(\mathbf{1}_\phi/s)\times\mathbf{A}_m(s,z)] = -(\mathbf{1}_\phi/s)\cdot\nabla\times\mathbf{A}_m = 0$$

$\mathrm{A}_m = -\nabla\psi$

rewritten in the form:

$$\overline{B}(s,z) = \overline{B}_a + \overline{B}_m, \qquad \overline{B}_a = \frac{\mathbf{1}_\phi}{s} s^2 J(s,z),$$
$$\overline{B}_m = -\frac{\mathbf{1}_\phi}{s} \times \nabla \psi \qquad \left(= \nabla \times \frac{\mathbf{1}_\phi}{s} \psi(s,z) \right). \tag{2.1}$$

For planets and steady stars the flow velocity continuity equation, $\partial \overline{\rho} / \partial t + \nabla \cdot \rho \overline{\mathrm{V}} = 0$, where $\overline{\rho}(\mathrm{r})$ is the the fluid density, takes a form: $\nabla \cdot \overline{\rho}(r) \overline{\mathrm{V}}_m = 0$. By analogy with the magnetic field we can write the flow velocity as

$$\overline{\mathrm{V}}(s,z) = \overline{\mathrm{V}}_a + \overline{\mathrm{V}}_m, \qquad \overline{\mathrm{V}}_a = \frac{\mathbf{1}_\phi}{s} s^2 \omega(s,z),$$
$$\overline{\mathrm{V}}_m = -\frac{\mathbf{1}_\phi}{s\overline{\rho}(r)} \times \nabla \chi \qquad \left(= \frac{1}{\overline{\rho}} \nabla \times \frac{\mathbf{1}_\phi}{s} \chi(s,z) \right). \tag{2.2}$$

To derive an equation for ψ let us apply to (1.2) operator opposite to curl: $[\nabla \times]^{-1}$ and average the equation obtained over ϕ. The potential part of the azimuthal component of electric field $\nabla \Psi$, arising under this operation, vanishes due to averaging over ϕ and so we obtain:

$$\frac{\partial}{\partial t} \frac{\mathbf{1}_\phi}{s} \psi = -\overline{\left[\frac{\mathbf{1}_\phi}{s} \frac{\partial}{\partial \phi} \Psi + \overline{\mathrm{E}}_a^s \right]} = -\frac{\overline{\mathrm{j}}_a}{\sigma} + \left[\overline{\mathrm{V} \times \mathrm{B}} \right]_a, \tag{2.3}$$

where $\overline{\mathrm{E}}^s$ is the solenoidal electrical field, $\overline{\mathrm{j}}_a / \sigma = -(\mathbf{1}_\phi / s) \eta \nabla \cdot s^2 \nabla (\psi / s^2)$ and

$$\left[\overline{\mathbf{V} \times \mathbf{B}} \right]_a = -\overline{\mathbf{V}} \times \frac{\mathbf{1}_\phi}{s} \times \nabla \psi + \left[\overline{\tilde{\mathbf{V}} \times \tilde{\mathbf{B}}} \right]_a = -\frac{\mathbf{1}_\phi}{s} \left(\overline{\mathbf{V}} \cdot \nabla \psi \right) + \overline{\tilde{\mathbf{V}}_m \times \tilde{\mathbf{B}}_m}.$$

Finally applying to (2.3) dot multiplication by $\mathbf{1}_\phi$ we obtain the equation for ψ:

$$\frac{\partial \psi}{\partial t} = -\overline{\mathbf{V}}_m \cdot \nabla \psi + s f_\phi + \eta \nabla \cdot s^2 \nabla \frac{\psi}{s^2}, \tag{2.4}$$

where electromotive force (emf), f_ϕ, equals

$$f_\phi = \mathbf{1}_\varphi \cdot \overline{\tilde{\mathbf{V}}_m \times \tilde{\mathbf{B}}_m} = \overline{\tilde{V}_z \tilde{B}_s} - \overline{\tilde{V}_s \tilde{B}_z} \qquad \left(= \overline{\tilde{V}_r \tilde{B}_\vartheta} - \overline{\tilde{V}_\vartheta \tilde{B}_r} \right) \tag{2.5}$$

Equation for toroidal field can be obtained by averaging of (1.9):

$$\frac{\partial J}{\partial t}=\nabla\cdot\left[\overline{\mathbf{B}}_m\omega-\overline{\mathbf{V}}_m J+\overline{\mathbf{f}}_m+\frac{\eta}{s^2}\nabla s^2 J\right],$$

$$\mathbf{f}_m=\frac{1}{s}\left[\overline{\tilde{\mathbf{B}}_m\tilde{V}_\phi}-\overline{\tilde{\mathbf{V}}_m\tilde{B}_\phi}\right].\qquad \text{where} \qquad (2.6,7)$$

Terms sf_ϕ and $\nabla\cdot\mathbf{f}_m$ describe the action of electromotive forces on the mean fields. In absence of these terms (2.4), (2.6) convert into well known equations for the axisymmetric magnetic field (see e.g. Moffatt 1978, (3.44), (3.43)).

Any physical quantity changes in any volume, occur due to quantity creation within the volume and due to flux of this quantity across the walls surrounding the volume. It is remarkable that the toroidal field has no sources in any volume. It changes only due to flux $\overline{\mathbf{B}}_m\omega-\overline{\mathbf{V}}_m J+\overline{\mathbf{f}}_m+\eta/s^2\nabla s^2 J$ of $J(s,z)$ across the walls of the volume.

Emf (2.5), (2.7) expressed by non-axisymmetric magnetic field, equations for which can be easily obtained from (1.9,10):

$$\frac{\partial}{\partial t}\frac{\tilde{B}_\phi}{s}=\nabla\tilde{\mathbf{F}}_m+\nabla\cdot\frac{\eta}{s^2}\nabla s^2\frac{\tilde{B}_\phi}{s}-\frac{\partial}{\partial\phi}\left(\tilde{\mathbf{B}}_m\cdot\nabla\frac{\eta}{s^2}\right), \qquad (2.8)$$

$$\frac{\partial\tilde{\mathbf{B}}_m}{\partial t}=-\frac{\partial}{\partial\phi}\tilde{\mathbf{F}}_m+\frac{\mathbf{1}_\phi}{s}\times\nabla s\tilde{F}_\phi-\left[\nabla\times\left[\eta\nabla\times\tilde{\mathbf{B}}\right]\right]_m, \qquad (2.9)$$

where

$$s\tilde{F}_\phi=\frac{\nabla\cdot\left[\tilde{\mathbf{B}}_m\chi-\overline{\rho}\tilde{\mathbf{V}}_m\psi\right]}{\overline{\rho}},\qquad \tilde{\mathbf{F}}_m=\overline{\mathbf{B}}_m\frac{\tilde{V}_\phi}{s}-\overline{\mathbf{V}}_m\frac{\tilde{B}_\phi}{s}+\tilde{\mathbf{B}}_m\omega-\tilde{\mathbf{V}}_m J. \qquad (2.10)$$

Equations (2.4) - (2.10) are equivalent to (1.9,10) but have clearer physical meaning. The axisymmetric magnetic field is supported by electromotive force created by the non-axisymmetric field. The latter is defined (2.8-10) depending on the axisymmetric field. If we are able to express this dependence explicitly we can convert the emf in the α effect.

Functions $\tilde{\mathbf{F}}_m$ and $\tilde{F}_\phi$ and emf $\tilde{\mathbf{f}}_m$ and $\tilde{f}_\phi$ being expressed through flow velocity are . The time derivatives in equations in (2.4,6) and (2.8,9) are of an order of \$1\$ as well as their diffusion terms out of layers. That is why we expect that emf and terms including velocity vanish in the leading approximation:

$$\tilde{\mathbf{f}}_m\sim\tilde{f}_\phi\sim 0,\qquad \tilde{\mathbf{F}}_m\sim\tilde{F}_\phi\sim 0,\qquad \overline{\mathbf{V}}_m\cdot\nabla\psi=-\left[\frac{\mathbf{1}_\phi}{\overline{\rho}s}\times\nabla\chi\right]\cdot\nabla\psi\sim 0, \qquad (2.11\text{-}14)$$

$$\overline{\mathbf{B}}_m\omega-\overline{\mathbf{V}}_m J\sim 0.$$

It is easy to check that the simplest possibility to satisfy these relations is to assume that

$$\tilde{\mathbf{B}} = k\overline{\rho}(r)\tilde{\mathbf{V}}(r,\vartheta,\phi), \qquad \psi = k\chi(r,\vartheta),$$
$$J = k\overline{\rho}(r)\omega(r,\vartheta), \tag{2.15,16,17}$$

where k is constant. If these relations do take place, they do not take place everywhere, but only in locations where the flow velocity is really large ($\sim R_m$).

As it was noted above our equations (2.4), (2.6) for axisymmetric part of magnetic field in absence of emf coincide with equations for axisymmetric magnetic field quoted by Moffatt. Nevertheless, the essential difference between these equations is that our equations are valid for compressible liquid whereas Moffatt's equations had been derived for incompressible one. If a liquid is incompressible, $\nabla\cdot\overline{\mathbf{V}}_m = 0$, then we can enter $\overline{\mathbf{V}}_m$ under sign ∇: $\overline{\mathbf{V}}_m\cdot\nabla\psi = \nabla\cdot\overline{\mathbf{V}}_m\psi$. In the case of a compressible liquid the continuity equation takes a form: $\nabla\cdot\overline{\mathbf{V}}_m = -\left(\overline{\rho}(r)'/\overline{\rho}(r)\right)\overline{V}_r$, where $\overline{\rho}(r)' = \partial\overline{\rho}(r)/\partial r$ and so an additional term $-\overline{V}_r\psi\partial\ln\overline{\rho}/\partial r$ arises in the equation. Of course the same difference takes place in the equation (2.6) for the toroidal field. To see that this difference is essential let us discuss the Cowling theorem.

Cowling (1934) proved that in the axisymmetric case steady solution is impossible. Braginsky's statement (1964) (see also section 6.4 in Moffatt (1978)) was more strong. He showed that axisymmetric field attenuates in presence of axisymmetric flow. However Braginsky's proof depends crucially on incompressibility of the liquid. To see this let us following Braginsky multiply (2.4) by ψ/η :

$$\frac{\partial}{\partial t}\frac{\psi^2}{2\eta} = -\nabla\cdot\overline{\mathbf{V}}_m\frac{\psi^2}{2\eta} - \frac{\overline{\rho'(r)}}{\overline{\rho}}\overline{V}_r\frac{\psi^2}{2\eta} + sf_\phi\frac{\psi}{\eta} + \psi\nabla\cdot s^2\nabla\frac{\psi}{s^2}$$

and integrate over whole space:

$$\frac{\partial}{\partial t}\int_{V\infty} d\mathbf{r}^3\frac{\psi^2}{2\eta} = -\int_{Sc} d\mathbf{S}\cdot\overline{\mathbf{V}}_m\frac{\psi^2}{2\eta} - \int_{Vc} d\mathbf{r}^3\frac{\overline{\rho'(r)}}{\overline{\rho}}\overline{V}_r\frac{\psi^2}{2\eta} +$$
$$\int_{Vc} d\mathbf{r}^3 sf_\phi\frac{\psi}{\eta} - \int_{V\infty} d\mathbf{r}^3\psi\nabla\cdot s^2\nabla\frac{\psi}{s^2}.$$

Out of the conducting region $\eta = \infty$ and so integral in lhs is limited. The first integral in rhs is over the surface bounding region where flow exists. This integral vanishes if $V_n=0$ on this surface. The second and the third integrals are over the volume bounded by this surface and so they are limited as well. The last integral is over unlimited volume:

$$\int_{V\infty} d\mathbf{r}^3\psi\nabla\cdot s^2\nabla\frac{\psi}{s^2} = \int_{V\infty} d\mathbf{r}^3\left[\nabla\cdot\mathbf{h}\psi - \mathbf{h}\cdot\nabla\psi\right] = -\int_{V\infty} d\mathbf{r}^3\left(\nabla\psi\right)^2,$$

$$\text{where}\qquad \mathbf{h} = \nabla\psi - 2\frac{\mathbf{1}_s\psi}{s}.$$

For magnetic field of dipole type $= 0\left(r^{-1}\right)$ at $r \to \infty$. That is why surface integrals vanish here and (2.11) becomes

$$\frac{\partial}{\partial t}\int_{V\infty} d\mathbf{r}^3 \frac{\psi^2}{2\eta} = -\int_{Vc} d\mathbf{r}^3 \frac{\overline{\rho'(r)}}{\overline{\rho}}\overline{V}_r \frac{\psi^2}{2\eta} + \int_{Vc} d\mathbf{r}^3 s f_\phi \frac{\psi}{\eta} - \int_{V\infty} d\mathbf{r}^3 (\nabla\psi)^2. \quad (2.18)$$

Eq. (2.18) shows that poloidal field is maintained by the compressibility and the effects (the first and the second terms in rhs of (2.18) respectively). If they are absent (for the former density have to be uniform: $\rho' = 0$) then $^2 \to 0$ and poloidal field attenuates. The role of ? effect is well known. Therefore, we will only discuss the compressibility effect further. In the second integral in rhs of (2.18) density gradient $\overline{\rho'}$ is negative, ψ^2 is positive, but V_r changes its sign depending on location. So this integral has no definite sign independent on flow. Additionally note that $\int V_r(r,\vartheta) r^2 \sin\vartheta d\vartheta d\phi = 0$ due continuity equation. That is why spherically symmetric part of $\psi, \psi = \psi(r)$, does not include in the discussing integral. It means that this integral does not vanish for those magnetic fields only where both components (B_r and B_ϑ) exist.

If flow is directed up in the regions where ψ^2 is big and is directed down where ψ^2 is small, then this integral would be positive and would prevent attenuation. It would enhance it in the opposite case. Is there any reason which makes this integral to be zero? In other words is there any reason forbidding compressible generation of poloidal field? It seems doubtful to me.

The physical meaning of this term supporting poloidal field is as follows. The magnetic field develops its own pressure. This pressure works over flow. If this work is negative it means that the flow transforms its energy into magnetic energy and thus maintains the magnetic field against dissipation.

One can say "Well, but even if we admit that this integral does not vanish, it can be too small to prevent attenuation of the poloidal field caused by the third term of (2.18)". To discuss this possibility let us convert (2.18) into dimensionless form using as usually typical space scale $\boldsymbol{L}_l$ as unit of space, $t_1 = L_1^2/\eta$ as unit of time and $V_1 = R_m\eta/L_1$ as unit of the flow velocity:

$$\frac{\partial}{\partial t}\int_{V\infty} d\mathbf{r}^3 \frac{\psi^2}{2} =$$

$$-R_m\int_{Vc} d\mathbf{r}^3 \frac{\overline{\rho'(r)}}{\overline{\rho}}\overline{V}_r \frac{\psi^2}{2} + R_m\int_{Vc} d\mathbf{r}^3 s f_\phi \psi - \int_{V\infty} d\mathbf{r}^3 (\nabla\psi)^2. \quad (2.18a)$$

In frame of the kinematic dynamo models the amplitudes of the flow ($\boldsymbol{R}_m$) can be arbitrarily large and so the same is valid for the compressible generating term in (2.18). In the Earth's geodynamo $\boldsymbol{R}_m$ is large, $\sim 10^3$ and it seems unlikely for the generation term to be smaller than the dissipation term for any type of flow.

Let us omit emf in (2.4), (2.6). Then taking into account definition of the substantial derivative

$$\frac{DA}{Dt}=\frac{\partial A}{\partial t}+\overline{\mathbf{V}}_m\cdot\nabla A$$

and the continuity equation $\nabla\cdot\overline{\mathbf{V}}_m=-\left(\overline{\rho}(r)'/\overline{\rho}(r)\right)\overline{V}_r$, we can write (2.4) in two different forms:

$$\frac{D\psi}{Dt}=\eta\nabla\cdot s^2\nabla\frac{\psi}{s^2},\qquad \frac{\partial\psi}{\partial t}=-\nabla\cdot\overline{\mathbf{V}}_m\psi+\frac{\overline{\rho'}(r)}{\overline{\rho}}\psi\overline{V}_r+\eta\nabla\cdot s^2\nabla\frac{\psi}{s^2}. \quad (2.19)$$

The first of them shows that ψ changes in a small liquid parcel only due to diffusion from it. Integration of the second one over a closed volume bounded by any surface $\chi(s,z)=const$ gives:

$$\frac{\partial}{\partial t}\iiint\psi d\mathbf{r}^3=\iiint d\mathbf{r}^3\frac{\overline{\rho'}(r)}{\overline{\rho}}\psi\overline{V}_r+\eta\iint s^2 d\mathbf{S}\cdot\nabla\frac{\psi}{s^2}.$$

It means that ψ in the volume changes due to its production by compressibility and diffusion from the volume. Hear we take into account that $\overline{\mathrm{V}}_m$ is orthogonal to surfaces$\chi(s,z)=const$

By analogy with (2.19) we obtain two equations for the toroidal field:

$$\begin{aligned}\frac{\partial J}{\partial t}&=\nabla\cdot\left[\overline{\mathrm{B}}_m\omega-\overline{\mathrm{V}}_m J\right]+\nabla\cdot\frac{\eta}{s^2}\nabla s^2 J,\\ \frac{DJ}{Dt}&=\overline{\mathrm{B}}_m\cdot\nabla\omega+\frac{\overline{\rho'}(r)}{\overline{\rho}}J\overline{V}_r+\nabla\cdot\frac{\eta}{s^2}\nabla s^2 J.\end{aligned} \quad (2.20)$$

The second equations shows that J changes in a small liquid parcel due to the ω and compressibility effects and due to diffusion from the parcel. Integration of the first over a closed volume gives:

$$\frac{\partial}{\partial t}\iiint J d\mathrm{r}^3=\iint d\mathrm{S}\cdot\overline{\mathrm{B}}_m\omega-\iint d\mathrm{S}\cdot\overline{\mathrm{V}}_m J+\iint d\mathrm{S}\cdot\frac{\eta}{s^2}\nabla s^2 J.$$

Thus J in any volume changes due to ω effect on its boundary, due to flux of J carried by the flow velocity across boundary and due to diffusion from the volume. If this volume is bounded by surface $\psi(s,z)=const$ then action of the ω effect over the volume is absent. Respectively flow transport of J across the boundary is absent for volumes bounded by surfaces $\chi=const$. Note that $\chi(r,\vartheta)=const$ on ICB and CMB due to non-penetrating condition $\mathrm{n}\cdot\mathrm{V}_m=0$.

Some words about adjusting conditions on the boundary with solid bodies. Magnetic field is continuous across the boundary. For the sake of convenience we adopt that

ψ is continuous as well. Additionally taking (1.16-18) into account we can summarize adjusting conditions for axisymmetrical magnetic field on the boundary:

$$\{\psi\}=0, \quad \{\nabla\psi\}=0, \quad \{J\}=0, \quad \left\{\eta\frac{\partial^2\psi}{\partial r^2}\right\}=0, \quad \left\{\eta\frac{\partial J}{\partial r}\right\}=0. \tag{2.21}$$

At axis $\boldsymbol{Z}$, regularity conditions apply, so that only solutions finite as $s\to 0$ are admitted. Therefore we demand that the mean over the circle with radius s meridional magnetic field, ψ/s^2, and the mean meridional electric current have to be finite:

$$= s^2 A(s,z), \qquad J = C(s,z),$$

where $\boldsymbol{A}$ and $\boldsymbol{C}$ are continuous and limited functions of z,s . Then near the axis only the z component of the mean magnetic field does not vanish:

$$\overline{\mathrm{B}}_m = -\frac{1_\phi}{s}\times\nabla s^2 A = 21_z A - s1_\phi\times\nabla A, \qquad B_\phi = sC(s,z).$$

In the solid core where (2.4,6) take the form

$$\frac{\partial\psi}{\partial t} = \eta\nabla\cdot s^2\nabla\frac{\psi}{s^2}, \qquad \frac{\partial J}{\partial t} = \nabla\cdot\frac{\eta}{s^2}\nabla s^2 J. \tag{2.22,23}$$

one can expect uniform $\boldsymbol{B}_z$ and zero circulation near the axis $\boldsymbol{Z}$. Glatzmaier and Roberts (1996b) numerical solution shows something like this expectation: $\boldsymbol{B}_z$ is approximately uniform near the axiz $\boldsymbol{Z}$.

Boundary layer

The induction equation for the axisymmetric poloidal field can be obtained directly by averaging (1.10) over ϕ. Let us transform the equation obtained and the equation for toroidal field (2.6) to dimensionless form using as usually typical space scale $\boldsymbol{L}_1$ as unit of space, $\mathbf{t}_1 = \boldsymbol{L}_1/\eta$ as unit of time and $\boldsymbol{V}_1 = \boldsymbol{R}_m\eta/\boldsymbol{L}_1$ as unit of the flow velocity

$$\frac{\partial\overline{\mathrm{B}}_m}{\partial t} = R_m\frac{1_\phi}{s}\times\nabla s\overline{F}_\phi + \left[\Delta\overline{\mathrm{B}}\right]_m, \qquad \overline{F}_\phi = 1_\phi\cdot\left[\overline{\mathrm{V}_m\times\mathrm{B}_m}\right] = \left[\overline{V_r B_\vartheta - V_\vartheta B_r}\right] \tag{3.1}$$

$$\frac{\partial J}{\partial t} = R_m\nabla\cdot\left[\overline{\mathrm{B}}_m\omega - \overline{\mathrm{V}}_m J + \overline{\mathrm{f}}_m\right] + \nabla\cdot\left[\frac{1}{s^2}\nabla s^2 J\right],$$
$$\mathrm{f}_m = \frac{1}{s}\left[\overline{\tilde{\mathrm{B}}_m\tilde{V}_\phi} - \overline{\tilde{\mathrm{V}}_m\tilde{B}_\phi}\right] \tag{3.2}$$

To satisfy these equation in the bulk of the core we have to assume that large terms vanish in the leading approximation:(3.2)

$$\overline{F}_{\phi} = o\left(R_m^{-1}\right) \qquad \nabla \cdot \left[\overline{\mathbf{B}}_m \omega - \overline{\mathbf{V}}_m J + \overline{\mathbf{f}}_m\right] = 0\left(R_m^{-1}\right) \tag{3.3}$$

However, in the boundary layers, where velocity changes sharply, these conditions disturb. That is why the large $\left(0\left(R_m\right)\right)\,\alpha$ and ω effects could be expected in the boundary layers. Is this layer generation essential for the field in the bulk of the core? This is the question to be discussed in this section.

A crude estimation based on angular moment conservation shows that the angular velocity of the mantle is much smaller than that of the inner core:

$$\frac{\omega^m}{\omega^{ic}} \sim -\frac{r_2^3 r_1^2}{r_3^5}\frac{\rho^{ic}}{\rho^m} \approx 0.015 \qquad \left(\frac{r_2^3 r_1^2}{r_3^5}\frac{\rho^{ic}}{\rho^m} \approx \frac{3.5^3 1.2^2}{6.4^5}\frac{13}{5}\right),$$

where $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}_3$ are the radiuses of the inner core, the outer core and the Earth respectively; ρ^{ic} and ρ^{m} are densities of the inner core and the mantle. Thus activity of the ICB boundary layer in the field generation has to be much larger than that of CMB. So further we will discuss just ICB layer having in mind that the layer on CMB can be treated in the same manner.

In the boundary layer the normal to the boundary component of velocity, V_r, is negligible. Tangential to the layer, derivatives are negligible as well in comparison with the normal one. Then the equations (3.1,2) take the form:

$$\frac{\partial \overline{\mathbf{B}}_m}{\partial t} = \mathbf{1}_{\vartheta} R_m \frac{\partial \overline{F}_{\phi}}{\partial r} + \frac{\partial^2 \overline{\mathbf{B}}_m}{\partial r^2}, \qquad \overline{F}_{\phi} = -\overline{V_{\vartheta} B_r} = -\overline{B}_r \overline{V}_{\vartheta} - \overline{\tilde{B}_r \tilde{V}_{\vartheta}}, \tag{3.4}$$

$$\frac{\partial J}{\partial t} = R_m \frac{\partial}{\partial r}\left[\overline{B}_r \omega + \overline{f}_r\right] + \frac{\partial^2 J}{\partial r^2}, \qquad f_r = \frac{1}{s}\overline{\tilde{B}_r \tilde{V}_{\phi}}. \tag{3.5}$$

The $\boldsymbol{B}_r$ component of the field does not change sharply in the layer. Respectively (3.4) describes, in fact, the creation of $\boldsymbol{B}_{\vartheta}$, by the shearing ($\overline{V}_{\vartheta}\overline{B}_r$) and the α ($\overline{\tilde{V}_{\vartheta}\tilde{B}_r}$) effects while is then defined by the continuity equation $\nabla \cdot \overline{\mathbf{B}}_m = 0$.

The time derivatives in (3.4,5) are of the order of one and so are negligible in comparison with the large α and ω effects and the diffusion. Then integrating (3.4,5) across the layer one obtain:

$$\left[\frac{\partial \overline{B}_{\vartheta}}{\partial r}\right] = R_m \overline{B}_r \left[\overline{V}_{\vartheta}\right] + R_m \left[\overline{\tilde{B}_r \tilde{V}_{\vartheta}}\right] \tag{3.6}$$

$$\left[\frac{\partial J}{\partial r}\right] = -R_m \overline{B}_r \left[\omega\right] - R_m \frac{1}{s}\left[\overline{\tilde{B}_r \tilde{V}_{\phi}}\right] \tag{3.7}$$

where square brackets means the jump across the layer. Eqs. (3.6,7) show that discontinuity of fluxes of B_{ϑ} and $\boldsymbol{J}$ is defined by production of these values in the layer just as heat sources create a discontinuity of a heat flux.

Thus the toroidal and the poloidal fields in the interior of the liquid core have two types of sources. The first one is the "volume" α and the ω effects in (3.1,2) occurring in the bulk of the core. The second type is the layers generation due to discontinuities of fluxes of $\overline{B}_\vartheta$ and $\boldsymbol{J}$ on the boundaries. Sources in the solid core and in the mantles are absent. The fields related to these effects can be present as a sum of two solutions. The first ones are governed by (3.1,2) and have no jumps of the normal derivatives on the boundaries. The second ones are obeyed in the liquid and solid core the diffusional equations

$$\frac{\partial \overline{B}_\vartheta}{\partial t} = \left[\Delta \overline{B}\right]_\vartheta \qquad \frac{\partial J}{\partial t} = \nabla \cdot \left[\frac{1}{s^2} \nabla s^2 J\right], \tag{3.8,9}$$

and satisfy the discontinues boundary conditions (3.6,7).

Which of these types of generation does prevail, the volume or the layer? Taking (3.6,7) into account one can see that relation between them ~ $R_m\hat{\delta}$ where $\hat{\delta}$ is dimensionless thickness of the layer. If the parameter $R_m\hat{\delta}$ is small then the influence of the layer generation is small as well and so only small disturbances could be expected on the boundaries. Further we will discuss just the opposite case when $R_m\hat{\delta} \geq 1$.

The toroidal field can be described by contours of constant $\boldsymbol{J}$. The case when $R_m\hat{\delta} \geq 1$ means that $\Delta\boldsymbol{J}$, the jump of $\boldsymbol{J}$ across the layer, is of order or even bigger than the typical values of $\boldsymbol{J}^0$ in the interior of the liquid core. Then the toroidal field is concentrated on the boundary and the most or the essential part of contours $\boldsymbol{J} = const$ are closed in the layer on ICB as this is depictured in Fig 1 A. This picture looks like the toroidal field distribution (Fig 3 D) by Glatzmaier and Roberts (1996 b). In the opposite case the lines of force are closed in the bulk of the liquid core.

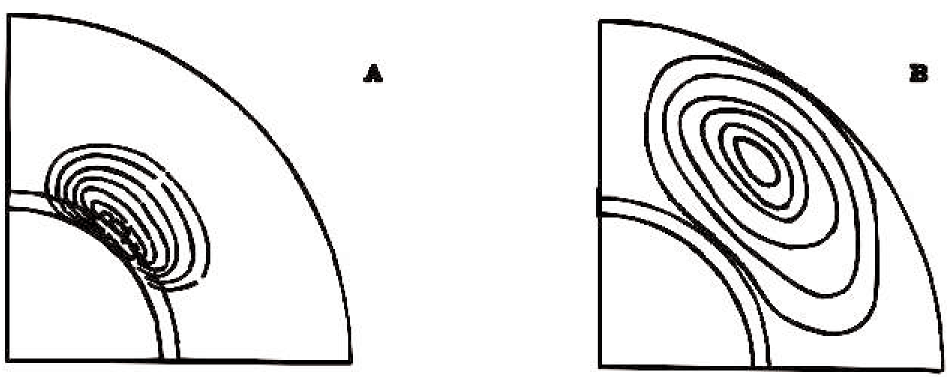

Fig. 1 Toroidal magnetic field. In the case $R_m\hat{\delta} \geq 1$ layers generation is large and so the most of contours of $\boldsymbol{B}\phi = const$ are mainly closed in the layer (**A**). If $R_m\hat{\delta} << 1$ (the Taylor state) they are mainly closed in the bulk of the liquid core (B).

Further we will discuss the poloidal field. What does the creation of $\boldsymbol{B}_\vartheta$ mean, taking into account that the "amount" of the creating in the layer field has to be zero due to the continuity equation? This means that the "positive" and "negative" values of $\boldsymbol{B}_\vartheta$ are separated in the layer and the "positive" values are transported to one its side and the "negative" one to the another, as it is pictured on Fig 2. Then $\overline{\boldsymbol{B}}_\vartheta$ diffuses into the liquid and

$\overline{B}_r$

the solid core. The picture of the poloidal field is completed when continuity equation $\nabla \cdot \mathrm{B}_m$ is taken into account. It creates B_r which connects the "positive" and the "negative" patterns of $\overline{B}_\vartheta$ into loops surrounded the boundary as this is depicted in Fig 3.

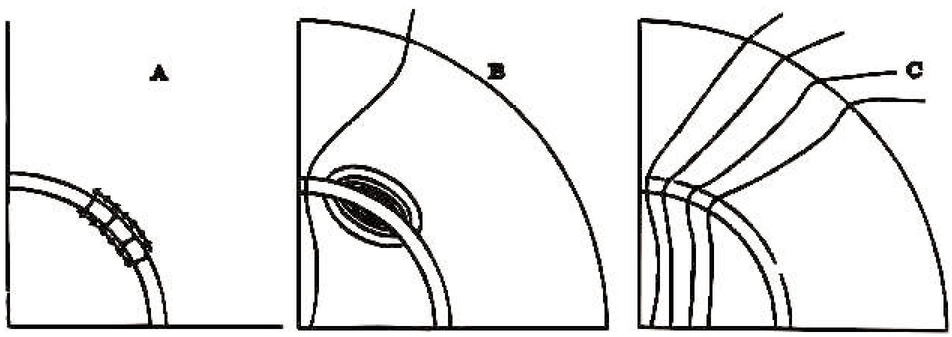

Fig. 2. Poloidal magnetic field. The "positive" and the "negative" portions of $\overline{B}_\vartheta$ are separated in the layer (A). Separated portions diffuse into the liquid and the solid core. B_r component is created taking the continuity equation $\nabla \cdot \mathrm{B}_m = 0$ into account forming closed loops $\psi = const$. In the case $R_m \hat{\delta} \geq 1$ layers generation dominate and the magnetic lines of force are mainly closed surrounding boundary (**B**). In the opposite case $R_m \hat{\delta} << 1$ (the Taylor state) they are only slightly disturbed at the boundaries (**C**).

The poloidal field can be treated as well in terms of stream function ψ satisfying in the case of the layer generation the equation $\partial \psi / \partial t = \nabla \cdot s^2 \nabla \psi / s^2$ in both the liquid and solid core. It can be shown that the contours of constant ψ form loops surrounding ICB and approximately symmetric about the surface $r=r_I=\delta/2$ as this depicted in Fig 3 A. This picture looks like the poloidal field distribution (Fig 3 C) obtained by Glatzmaier and Roberts (1996 b) computer simulation.

The contour lines of the toroidal field in the case $R_m \hat{\delta} \geq 1$ are mainly closed in the boundary layer as it is depicted at the Fig 3 B. This Figure can be compared with Fig 3 D obtained by computer simulations of by Glatzmaier and Roberts (1996 b).

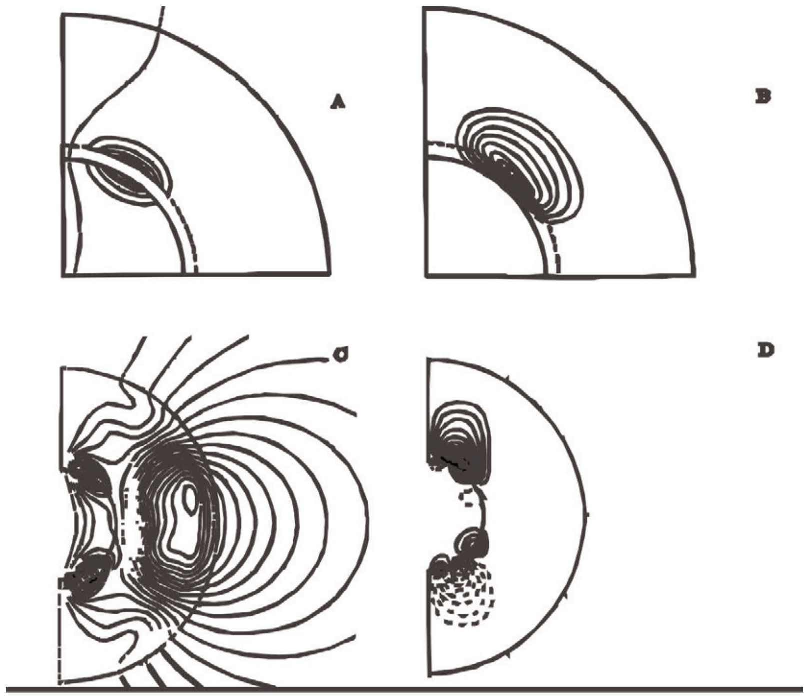

Fig. 3. Comparison of our results for the poloidal (**A**) and the toroidal (**B**) magnetic fields in the case $R_m\hat{\delta} \geq 1$ with Glatzmaier and Roberts (1996 b) numerical simulation **C** and **D** respectively. The lines of force of poloidal field in the simulation surround ICB just as in 3 A. The contours of $\mathbf{B}\ \phi$ on 3 D are mainly closed in the layer as in 3 B. Thus we conclude that these fields are of the layer origin.

Thus we conclude that parameter $R_m\hat{\delta}$ is very important for the generation of both the poloidal and the toroidal axisymmetric magnetic field. If this parameter is small then the influence of the layer generation on the entire generation in the core is negligible. In the opposite case it can be dominant. To estimate value of this parameter we need take into account the thickness of Ekman-Hartmann layer:

$$\hat{\delta} \sim \sqrt{\frac{2E}{\sqrt{\cos^2\vartheta + \hat{B}_r^4} + \hat{B}_r^2}}, \text{where } \hat{B}_r^2 = \frac{\overline{B}_r^2}{2\Omega\rho\eta\mu_o} \text{ and } E = \frac{\nu}{2\Omega L_1^2} \qquad (3.10)$$

If $\hat{B}_r^2 \leq 1$ on the boundary then $\hat{\delta}$ is of order of the thickness of the Ekmann layer $\hat{\delta} \approx \sqrt{E}$ but if $\hat{B}_r^2 >> 1$ then $\hat{\delta} \approx \sqrt{E}\Big/\left|\hat{B}_r\right|$. In any case $R_m\hat{\delta} \leq R_m\sqrt{E}$ Therefore, in the Taylor state, the solution at $E \to 0$, the parameter $R_m\hat{\delta}$ tends to zero. It means that the layer generation is not essential for the Taylor state and respectively the axisymmetric toroidal magnetic field is continuous at the boundaries in this limit. The axisymmetric poloidal field has no loops surrounding the boundary.

Does the Earth's magnetic field satisfy this condition for Taylor state? To answer let us estimate the value of the parameter $R_m\sqrt{E}$. For values of the kinematics viscosity typical for the liquid core the Ekman number is extremely small $E \sim 10^{-15}$ and so $R_m\hat{\delta}$ is small as well: $R_m\hat{\delta} \sim 10^{-5}$ (we accept here $R_m \sim 10^3$). The Ekman number essentially increases if we adopt that eddy diffusivity "works" in the core instead of kinematics one. If this turbulent viscosity equals the magnetic diffusivity, $\nu=\eta=2^{m}2^{s-1}$, then $\hat{\delta} = 5\times10^{-5}$ and $R_m\sqrt{E} \sim 5\times10^{-2}$. Since $R_m\hat{\delta} \le R_m\sqrt{E}$, we conclude that $R_m\hat{\delta} << 1$ and so the axisymmetric magnetic field satisfies the Taylor state condition in the Earth liquid core.

Are the existing numerical solutions of the problem satisfy too the condition for the Taylor state? Resolution of the thin boundary layer ~ $\sqrt{E}$ demands very powerful computer resources. That is why Glatzmaier and Roberts (1996a,b) using a powerful Cray computer were able to solve the generation problem with a relatively large Ekman number: $E = 10^{-6}$ But even this value is not real due to the hyperdiffusivity used by these authors. We believe that the effective Ekman number in Glatzmaier and Roberts (1996b) simulations is of order of 10^{-4}. Their R_m at ICB estimated by the angular velocity of the inner core $\sim 10^3$. Then $R_m\sqrt{E}$ parameter is of order of . If the boundary layer is the Ekman one ($\hat{\delta} \sim \sqrt{E}$) then $R_m\hat{\delta} \le R_m\sqrt{E}$ and so the condition for Taylor state, $R_m\hat{\delta} << 1$, does not satisfy.

However, if the magnetic field at the boundary is large ($\hat{B}_r^2 >> 1$) then the thickness of the layer decreases: $\hat{\delta} \approx \sqrt{E}/|\hat{B}_r|$. Our estimate gives $\hat{B}_r^2 \sim R_m$ in the bulk of the core. If this is valid for the boundaries as well, then parameter $R_m\hat{\delta}$ becomes $\sqrt{R_m E}$ instead of $R_m\sqrt{E}$. Then parameter $R_m\hat{\delta}$ in Glatzmaier and Roberts solution is small . In fact it can be even smaller if we take into account that the jumps in (3.6,7), V_ϑ and ω are not necessarily of order of 1, but can be smaller.

That is why we cannot say definitely if $R_m\hat{\delta}$ small in the Glatzmaier and Roberts solution or not. To reveal whether the layer generation is essential in it let us have a look at their pictures of both of the field. On Fig 4 are shown the poloidal, Fig 4a, and the toroidal, Fig 4b, fields from GR (1996b). The lines of force of the poloidal field are localized mainly at ICB and surround this boundary. Furthermore, they are almost symmetric in relation to the surface which is slightly higher that r_1. This just coincides with our expectation for the case $R_m\hat{\delta} \ge 1$ that lines of force surround the boundary being approximately symmetric related to the surface $r = r_1 + \delta/2$. We expected also for this case that the contours of the constant toroidal field are condensed in the boundary layer. Fig 4b shows just this behavior of the field. Therefore we conclude that the axisymmetric field in Glatzmaier and Roberts (1996b) are mainly of the layer origin.

However, the same solution discussing by Glatzmaier and Roberts (2000) at another time period shows different behavior of the poloidal field. Their lines of force are have no loops surrounding ICB and are only slightly disturbed at the boundaries. We believe that the layer generation is not essential for the solution during the time period being under consideration. (It appears to be essential during the inversion discussing in this paper. Then loops surrounding ICB arise).

Thus we believe that the Ekman number in the Glatzmaier and Roberts simulation is small enough to arise the solution of the Taylor state type. However, it is still big to prevent arising of non-Taylor state behavior in another time periods.

Discussion

In this section we make a short resume of the results of our work. The magnetic field, $\mathbf{B}$, and flow velocity, $\mathbf{V}$, were decomposed on azimuthal, $\mathbf{B}_a = \mathbf{1}_\phi B_\phi$, $\mathbf{V}_a = \mathbf{1}_\phi V_\phi$ and meridional components, $\mathbf{B}_m = \mathbf{B}-\mathbf{B}_a$, $\mathbf{V}_m = \mathbf{V}-\mathbf{V}_a$ and both of them were separated into axisymmetric and non-axisymmetric parts.

Induction equations for the axisymmetric magnetic field, $\overline{\mathbf{B}} = \overline{\mathbf{B}}_a + \overline{\mathbf{B}}_m$ are obtained in the form:

$$\frac{\partial \psi}{\partial t} = -\overline{\mathbf{V}}_m \cdot \nabla \psi + s f_\phi + \eta \nabla \cdot s^2 \nabla \frac{\psi}{s^2} \qquad \mathbf{B}_m = -\frac{\mathbf{1}_\phi}{s} \times \nabla \psi(s,z) \tag{4.1}$$

$$\frac{\partial J}{\partial t} = \nabla \cdot \left[\overline{\mathbf{B}}_m \omega - \overline{\mathbf{V}}_m J + \overline{\mathbf{f}}_m + \frac{\eta}{s^2} \nabla s^2 J \right] \qquad \mathbf{B}_a = \mathbf{1}_\phi s J(s,z) \tag{4.2}$$

where the axisymmetric velocity is defined as

$$\overline{\mathbf{V}}(s,z) = \overline{\mathbf{V}}_a + \overline{\mathbf{V}}_m, \qquad \overline{\mathbf{V}}_a = \frac{\mathbf{1}_\phi}{s} s^2 \omega(s,z), \qquad \overline{\mathbf{V}}_m = -\frac{\mathbf{1}_\phi}{s\overline{\rho}(r)} \times \nabla \chi(s,z)$$

and $\overline{\rho}(r)$ is the liquid density. The toroidal field $\mathbf{B}_a$ is maintained by the ω effect (the first term in rhs of (4.2)) and both the toroidal and the poloidal fields are supported additionally by electromotive force (emf):

$$f_\phi = \mathbf{1}_\varphi \cdot \overline{\tilde{\mathbf{V}}_m \times \tilde{\mathbf{B}}_m}, \qquad \mathbf{f}_m = \frac{1}{s} \left[\overline{\tilde{\mathbf{B}}_m \tilde{V}_\phi} - \overline{\tilde{\mathbf{V}}_m \tilde{B}_\phi} \right] \tag{4.3}$$

where $\tilde{\mathbf{V}}$ is the non-axisymmetric flow velocity and $\tilde{\mathbf{B}}$ is the non-axisymmetric magnetic field which obeys by the equations:

$$\frac{\partial}{\partial t} \frac{\tilde{B}_\phi}{s} = \nabla \cdot \tilde{\mathbf{F}}_m + \nabla \cdot \frac{\eta}{s^2} \nabla s^2 \frac{\tilde{B}_\phi}{s} - \frac{\partial}{\partial \phi} \left(\tilde{\mathbf{B}} \cdot \nabla \frac{\eta}{s^2} \right), \tag{4.4}$$

$$\frac{\partial \tilde{\mathbf{B}}_m}{\partial t} = -\frac{\partial \tilde{\mathbf{F}}_m}{\partial \phi} + \frac{\mathbf{1}_\phi}{s} \times \nabla s \tilde{F}_\phi - \left[\nabla \times \left[\eta \nabla \times \tilde{\mathbf{B}} \right] \right]_m. \tag{4.5}$$

Here

$$s\tilde{F}_\phi = \frac{\nabla \cdot \left[\tilde{\mathbf{B}}_m \chi - \overline{\rho} \tilde{\mathbf{V}}_m \psi \right]}{\overline{\rho}}, \qquad \tilde{\mathbf{F}}_m = \overline{\mathbf{B}}_m \frac{\tilde{V}_\phi}{s} - \overline{\mathbf{V}}_m \frac{\tilde{B}_\phi}{s} + \tilde{\mathbf{B}}_m \omega - \tilde{\mathbf{V}}_m J \tag{4.6}$$

It is commonly assumed that the axisymmetric part of the poloidal magnetic field is only created by the α effect (by emf). However, there is one more generating effect implicitly included in (4.1) when liquid is compressible. It can be revealed by the following speculations. In case of an incompressible liquid we can enter $\overline{\mathbf{V}}_m$ under operator ∇ in the first term of rhs of (4.1): $\overline{\mathbf{V}}_m \cdot \nabla \psi = \nabla \cdot \overline{\mathbf{V}}_m \psi$, without changing other parts of the equation. In the case of a compressible liquid the same operation leads to the appearance of a new term in the equation: $\overline{\mathbf{V}}_m \cdot \nabla \psi = \nabla \cdot \overline{\mathbf{V}}_m \psi - \psi \nabla \cdot \overline{\mathbf{V}}_m = \nabla \cdot \overline{\mathbf{V}}_m \psi + \psi \overline{V}_r \left(\overline{\rho}' / \overline{\rho} \right)$, where $\overline{\rho}' = \partial \overline{\rho} / \partial r$. Here we take into account the continuity equation for the axisymmetric field: $\nabla \cdot \overline{\rho}(r) \overline{\mathbf{V}}_m = \overline{\rho}(r) \nabla \cdot \overline{\mathbf{V}} + \overline{V}_r \nabla \overline{\rho} = 0$.

This new term disturbs Braginsky's proof of the Cowling theorem: it appears that in the presence of this term value ψ^2 in general does not attenuate even in the absence of emf. This statement is not in conflict with Cowling's original proof, who has shown that a steady solution is impossible. A steady solution is impossible, but the non-steady ones could be possible!

Thus, we assume that the compressibility effect could play the same essential role in a generation of magnetic field as the α and the ω effects. Therefore, computer simulations neglecting the compressibility of the liquid lose an important source of the generation. We believe also that this new generating term (and analogous one for the toroidal field) could be especially important in stars where the density in the convective zone changes by a few orders.

A remarkable consequence of the new generating term is the possibility for a field generation in the pure axisymmetric case without α effect, which is commonly believed to be impossible due to the taboo of Cowling theorem.

It is usually assumed that magnetic field changes on the diffusion time scale $\tau = L_1^2 / \eta$, where L_1 is the space scale of the generating region. However, the time derivatives on the lhs of the equations (4.1,2) and (4.4,5) are defined on the rhs by the large ($0(R_m)$) terms: $\overline{\mathrm{V}}_m \cdot \nabla \psi$, $\nabla \cdot \left[\overline{\mathrm{B}}_m \omega - \overline{\mathrm{V}}_m J, \right] f_\phi$, f_m and $\tilde{F}_\phi$, $\tilde{\mathrm{F}}_m$, responding to a much shorter advective time scale $\sim \tau / R_m$. To avoid the contradiction, we have to assume that in the leading approximation these terms vanish in the regions where $R_m >> 1$. This is possible if e.g. the magnetic field satisfies the relations:

$$\tilde{\mathrm{B}} = k \overline{\rho}(r) \tilde{\mathrm{V}}(r, \vartheta, \phi) \qquad \psi = k \chi(r, \vartheta) \qquad J = k \overline{\rho}(r) \omega(r, \vartheta) \tag{4.7}$$

where $k = const$.

This hypothesis can be easily checked using the existing numerical solutions (e.g. that of Glatzmaier and Roberts). If it is confirmed this will expand our understanding of how generation "works".

In the boundary layers velocity changes sharply and the conditions leading to vanishing of the emf and effects are violated. That is why large ($0(R_m)$) emf and ω effects appear in the layers. Though the densities of the emf and ω effects are large $0(R_m)$ they are in the thin ($\hat{\delta} << 1$) layer and so their summary action could be small ($\sim R_m \hat{\delta} \le R_m \sqrt{E}$) and vanishing in the Taylor state limit $E \to 0$. However, their influence could be dominant in the computer simulations if the Ekman number is not small enough: $R_m \hat{\delta} \ge 1$.

What are the values of the parameter $R_m \hat{\delta}$ in the recent computer simulations? If we adopt that the boundary layer is the pure Ekman one, then in the Glatzmaier and Roberts (1996) solution $R_m \hat{\delta} \sim R_m \sqrt{E} \sim 10$. In order to decrease this parameter with two orders we must decrease E by four orders. That is why on the case of the Ekman layer the prognosis is not very optimistic. However, if the normal to the boundary component of the magnetic field is large enough ($\hat{B}_r^2 >> 1$, $\hat{B}_r^2(r,\vartheta) = B_r^2 / 2\Omega\rho\eta\mu_o$), then the thickness of the layer decreases: $\hat{\delta} \sim \sqrt{E} / \hat{B}$ and the parameter $R_m\hat{\delta}$ decreases as well: $R_m\hat{\delta} \sim R_m\sqrt{E} / \hat{B}$. Our estimate of the magnetic field gives $\hat{B}^2 \sim R_m$ in the bulk of the core. If this is valid for the boundaries as well, then parameter $R_m\hat{\delta}$ becomes $\sqrt{R_m E}$ instead of $R_m\sqrt{E}$. Then the parameter $R_m\hat{\delta}$ in the Glatzmaier and Roberts (1996b) solution is relatively small: ~ 0.3 .

Respectively the form of the axisymmetric magnetic field in the Glatzmaier and Roberts computer simulations in some moments is similar the layer generated field (Fig 1, 3), but in other moments it is similar to the Taylor state solution with small disturbances at the boundaries. We believe that a decrease of the Ekman number with one order can make their solution to be the Taylor state.

Appendix: Azimuthal and Meridional Decomposition

Any vector field $\mathbf{B}$ can be decomposed in the azimuthal, $\mathbf{B}_a = 1_\phi B_\phi$, and the meridional, $\mathbf{B}_m = \mathbf{B} - \mathbf{B}_a = -\mathbf{1}_\phi \times \mathbf{1}_\phi \times \mathbf{B}$ components:

$$\mathbf{B} = \mathbf{B}_a + \mathbf{B}_m \qquad \mathbf{B}_a = \frac{\mathbf{1}_\phi}{s} sB_\phi \qquad \mathbf{B}_m = \frac{\mathbf{1}_\phi}{s} \times \mathbf{A}_m \qquad \mathbf{A}_m = -\mathbf{1}_\phi \times s\mathbf{B} \tag{A1}$$

To obtain the same decomposition for $\mathbf{j} = \nabla \times \mathbf{B}$ we use standard formulas of vector analysis:

$$\nabla \times (\Phi \mathbf{F}) = \Phi \nabla \times \mathbf{F} + \nabla \Phi \times \mathbf{F}$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F})$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \qquad \nabla \cdot (\Phi \mathbf{F}) = \Phi \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla \Phi$$

is easy to check by direct evaluation that

$$\nabla \cdot \frac{\mathbf{1}_\phi}{s} = 0, \qquad \nabla \times \frac{\mathbf{1}_\phi}{s} = 0, \qquad \nabla \cdot \frac{\mathbf{1}_s}{s} = 0, \qquad \nabla \times \frac{\mathbf{1}_s}{s} = 0.$$

Then

$$\nabla \times \mathbf{B}_a = \nabla \times \frac{\mathbf{1}_\phi}{s} sB_\phi = sB_\phi \nabla \times \frac{\mathbf{1}_\phi}{s} - \frac{\mathbf{1}_\phi}{s} \times \nabla sB_\phi = -\frac{\mathbf{1}_\phi}{s} \times \nabla sB_\phi$$

$$\nabla \times \mathbf{B}_m = \nabla \times \frac{\mathbf{1}_\phi}{s} \times \mathbf{A}_m = (\mathbf{A}_m \cdot \nabla)\frac{\mathbf{1}_\phi}{s} - \left(\frac{\mathbf{1}_\phi}{s} \cdot \nabla\right)\mathbf{A}_m + \frac{\mathbf{1}_\phi}{s} \nabla \cdot \mathbf{A}_m =$$

$$\frac{1}{s^2}\left[-2\mathbf{1}_\phi A_s - \frac{\partial \mathbf{A}_m}{\partial \phi}\right] + \frac{\mathbf{1}_\phi}{s} \nabla \cdot \mathbf{A}_m = \frac{\partial \mathbf{1}_\phi \times \mathbf{B}}{s \partial \phi} + \mathbf{1}_\phi s \nabla \cdot \frac{\mathbf{A}_m}{s^2} = \frac{\mathbf{1}_\phi}{s} \times \frac{\partial \mathbf{B}_m}{\partial \phi} -$$

$$\frac{\mathbf{1}_\phi}{s} s^2 \nabla \cdot \frac{\mathbf{1}_\phi \times \mathbf{B}_m}{s} \qquad \left(= \frac{\mathbf{1}_\phi}{s} \times \frac{\partial \mathbf{B}_m}{\partial \phi} + \mathbf{1}_\phi (\mathbf{1}_\phi \cdot \nabla \times \mathbf{B}_m) \right)$$

Here we have taken into account that

$$(\mathbf{A}_m\cdot\nabla)\frac{\mathbf{1}_\phi}{s}-\left(\frac{\mathbf{1}_\phi}{s}\cdot\nabla\right)\mathbf{A}_m=sA_s\left[\left(\frac{\mathbf{1}_s}{s}\cdot\nabla\right)\frac{\mathbf{1}_\phi}{s}-\left(\frac{\mathbf{1}_\phi}{s}\cdot\nabla\right)\frac{\mathbf{1}_s}{s}\right]-\frac{1}{s^2}\frac{\partial\mathbf{A}_m}{\partial\phi}=$$
$$\frac{1}{s^2}\left[-2\mathbf{1}_\phi A_s-\frac{\partial\mathbf{A}_m}{\partial\phi}\right]$$

and

$$\left(\frac{\mathbf{1}_s}{s}\cdot\nabla\right)\frac{\mathbf{1}_\phi}{s}-\left(\frac{\mathbf{1}_\phi}{s}\cdot\nabla\right)\frac{\mathbf{1}_s}{s}=\nabla\times\left[\frac{\mathbf{1}_\phi}{s}\times\frac{\mathbf{1}_s}{s}\right]=-\nabla\times\frac{\mathbf{1}_z}{s^2}=-2\mathbf{1}_z\times\frac{\mathbf{1}_s}{s^3}=-2\frac{\mathbf{1}_\phi}{s^3}$$

Thus curl of azimuthal field $\mathbf{B}_a$ is pure meridional, but curl of meridional field is not pure azimuthal. It contents a meridional component vanishing under averaging over ϕ. Using these formulas the field $\mathbf{j}=\nabla\times\mathbf{B}$ can be presented as a sum of the azimuthal, $\mathbf{j}_a$, and the meridional, $\mathbf{j}_m$, components where

$$\mathbf{j}_a=\mathbf{1}_\phi(\mathbf{1}_\phi\cdot\nabla\times\mathbf{B}_m)=-\mathbf{1}_\phi s\nabla\cdot\left[\frac{\mathbf{1}_\phi}{s}\times\mathbf{B}_m\right]\qquad \mathbf{j}_m=\frac{\mathbf{1}_\phi}{s}\times\left[\frac{\partial\mathbf{B}_m}{\partial\phi}-\nabla sB_\phi\right]\tag{A3}$$

The vector production of two fields also can be decomposed in azimuthal meridional components:

$$\mathbf{V}\times\mathbf{B}=[\mathbf{1}_\phi V_\phi+\mathbf{V}_m]\times[\mathbf{1}_\phi B_\phi+\mathbf{B}_m]=\mathbf{1}_\phi\times[\mathbf{B}_mV_\phi-\mathbf{V}_mB_\phi]+\mathbf{V}_m\times\mathbf{B}_m$$

The meridional part of this vector so is

$$[\mathbf{V}\times\mathbf{B}]_m=\mathbf{1}_\phi\times s\mathbf{F}_m,$$
$$where\qquad s\mathbf{F}_m=-\mathbf{1}_\phi\times[\mathbf{V}\times\mathbf{B}]=\mathbf{B}_mV_\phi-\mathbf{V}_mB_\phi,\tag{A4}$$

and its azimuthal part takes a form:

$$[\mathbf{V}\times\mathbf{B}]_a=\mathbf{1}_\phi F_\phi$$
$$where\qquad F_\phi=\mathbf{1}_\phi\cdot[\mathbf{V}\times\mathbf{B}]=\mathbf{1}_\phi\cdot[\mathbf{V}_m\times\mathbf{B}_m]=V_zB_s-V_sB_z\tag{A5}$$

Induction equation contents field $\mathbf{d}=\nabla\times\eta\mathbf{j}$ which also have to be presented as a sum of azimuthal, $\mathbf{d}_a$, and the meridional, $\mathbf{d}_m$, components. Its azimuthal component can be easy evaluated using (A3):

$$d_\phi=-s\nabla\cdot\left[\frac{\mathbf{1}_\phi}{s}\times\eta\mathbf{j}_m\right]=s\nabla\cdot\frac{\eta}{s^2}\left[\frac{\partial\mathbf{B}_m}{\partial\phi}-\nabla_m sB_\phi\right]=s\nabla\frac{\eta}{s^2}\cdot\frac{\partial\mathbf{B}_m}{\partial\phi}+\frac{\eta}{s}\frac{\partial\nabla\cdot\mathbf{B}_m}{\partial\phi}-$$
$$s\nabla\cdot\frac{\eta}{s^2}\nabla m^S B_\phi=\frac{\partial}{\partial\phi}s\mathbf{B}_m\cdot\nabla\frac{\eta}{s^2}-\frac{\eta}{s}\frac{\partial^2B_\phi}{\partial\phi^2}-s\nabla\cdot\frac{\eta}{s^2}\nabla_m sB_\phi=$$

$$\frac{\partial}{\partial\phi} s\mathbf{B}\cdot\nabla\frac{\eta}{s^2} - s\frac{\partial}{\partial\phi}\frac{\eta}{s^2}\frac{\partial B_\phi}{\partial\phi} - s\nabla\cdot\frac{\eta}{s^2}\nabla_m sB_\phi$$

Thus

$$[\nabla\times\eta\mathbf{j}]_\phi = -s\left[\nabla\cdot\frac{\eta}{s^2}\nabla s^2\frac{B_\phi}{s} - \frac{\partial}{\partial\phi}\left(\mathbf{B}\cdot\nabla\frac{\eta}{s^2}\right)\right] \tag{A6}$$

Here we used the continuity equation for magnetic field: $(\partial B_\phi/\partial\phi) = -s\nabla\cdot\mathbf{B}_m$
The meridional component of $\mathbf{d}$ in case of non-uniform η generally speaking has no so simple form. That is why further we adopt that η is uniform in the liquid core and can change only at boundaries (on ICB and CMB). Then

$$\mathbf{d} = \nabla\times\eta\nabla\times\mathbf{B} = \eta\nabla\times\nabla\times\mathbf{B} = \eta\nabla\nabla\cdot\mathbf{B} - \eta\nabla^2\mathbf{B} = -\eta\nabla^2\mathbf{B}$$

where $\nabla^2\mathbf{B}$ in the polar coordinates (s,ϕ,z) takes a form:

$$\nabla^2\mathbf{B} = \mathbf{1}_s\left[\Delta B_s - \frac{B_s}{s^2} - \frac{2}{s^2}\frac{\partial B_\phi}{\partial\phi}\right] + \mathbf{1}_\phi\left[\Delta B_\phi - \frac{B_\phi}{s^2} - \frac{2}{s^2}\frac{\partial B_s}{\partial\phi}\right] + \mathbf{1}_z\Delta B_z \tag{A7}$$

Here we have taken into account the continuity equation for magnetic field $\nabla\cdot\mathbf{B}=0$.
Using the identities

$$s^{-1}\nabla\cdot s^2\nabla s^{-1}A = s\nabla\cdot s^{-2}\nabla sA \equiv \Delta A - \frac{A}{s^2}$$

which easily can be checked directly:

$$s^{-1}\nabla\cdot s^2\nabla s^{-1}A = s^{-1}\nabla\cdot[-\mathbf{1}_s A + s\nabla A] = -s^{-2}\frac{\partial sA}{\partial s} + s^{-1}\frac{\partial A}{\partial s} + \Delta A = \Delta A - \frac{A}{s^2}$$

$$s\nabla\cdot s^{-2}\nabla sA = s\nabla\cdot[s^{-2}\mathbf{1}_s A + s^{-1}\nabla A] = \frac{\partial s^{-1}A}{\partial s} - s^{-1}\frac{\partial A}{\partial s} + \Delta A = \Delta A - \frac{A}{s^2}$$

Now we can rewrite (A7) in a form:

$$\nabla^2\mathbf{B} = \mathbf{1}_s\left[s^{-1}\nabla\cdot s^2\nabla s^{-1}B_s - \frac{2}{s^2}\frac{\partial B_\phi}{\partial\phi}\right] +$$

$$\mathbf{1}_\phi\left[s\nabla\cdot s^{-2}\nabla sB_\phi + \frac{2}{s^2}\frac{\partial B_s}{\partial\phi}\right] + \mathbf{1}_z\Delta B_z \tag{A8}$$

Using (A1) and (A3) we can decompose the Lorentz force $\mathbf{B}\times\mathbf{j}$:

$$\mathbf{B}\times\mathbf{j} = [\mathbf{B}_a + \mathbf{B}_m]\times[\mathbf{j}_a + \mathbf{j}_m] = \mathbf{B}_a\times\mathbf{j}_m + \mathbf{B}_m\times\mathbf{j}_a + \mathbf{B}_m\times\mathbf{j}_m$$

where

$$\mathbf{B}_a \times \mathbf{j}_m = -\frac{1}{s}\frac{\partial B_\phi \mathbf{B}_m}{\partial \phi} - \mathbf{B}_m \nabla \cdot \mathbf{B}_m + \frac{1}{s^2}\nabla \frac{s^2 B_\phi^2}{2},$$

$$\mathbf{B}_m \times \mathbf{j}_a = \mathbf{1}_\phi \times \mathbf{B}_m s \nabla \cdot \left[\frac{\mathbf{1}_\phi}{s} \times \mathbf{B}_m\right],$$

$$\mathbf{B}_m \times \mathbf{j}_m = \frac{\mathbf{1}_\phi}{s}\left[\frac{\partial}{\partial \phi}\frac{\mathbf{B}_m^2 - B_\phi^2}{2} - \nabla \cdot \mathbf{B}_m s B_\phi\right].$$

Respectively

$$[\mathbf{B} \times \mathbf{j}]_a = \frac{\mathbf{1}_\phi}{s}\left[\frac{\partial}{\partial \phi}\frac{\mathbf{B}_m^2 - B_\phi^2}{2} - \nabla \cdot \mathbf{B}_m s B_\phi\right] \tag{A9}$$

$$[\mathbf{B} \times \mathbf{j}]_m = -\frac{1}{s}\frac{\partial B_\phi \mathbf{B}_m}{\partial \phi} + \frac{1}{s^2}\nabla_m \frac{s^2 B_\phi^2}{2} -$$

$$\mathbf{B}_m \nabla \cdot \mathbf{B}_m + \mathbf{1}_\phi \times \mathbf{B}_m s \nabla \cdot \left[\frac{\mathbf{1}_\phi}{s} \times \mathbf{B}_m\right] \tag{A10}$$

References

Braginsky, S.I. 1964. Self-exitation of magnetic field during the motion of a highly conducting fluid, *Zh. Eksp. Teor. Fiz. SSSR*, **47**, 1084-1098 [Sov.Phys. JETP 20, 726-735].

Cowling, T.G. 1934. The magnetic field of sunspots, *Mon. Not. Roy. Astr. Soc.*, **94**, 39-48.

Glatzmaier, G.A., and P.H. Roberts, 1995a. A three-dimensional convective dynamo solution with rotating and finitely conducting inner core and mantle, *Phys. Earth Planet. Inter.*, **91**, 63-75.

Glatzmaier, G.A., and P.H. Roberts, 1996a. An anelastic evolutionary geodynamo simulation driven by compositional and thermal convection, *Physica*, **D97,** 81-94.

Glatzmaier, G.A., and P.H. Roberts, 1996b. Rotation and Magnetism of Earth's Inner Core, *Science*, **274**, 1887-1891.

Roberts, P.H., and G.A. Glatzmaier, 2000. Geodynamo theory and simulations, *Rew. of Modern Physics*, **72**, 1081-1123.

Moffatt, H.K., 1978. *Magnetic Field Generation in Electrically Conducting Fluids*. Cambridge UK: University Press.

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Ролята на граничен α -ефект в генерацията на геомагнитното поле

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Резюме: Въртящите се звезди и планети притежават цилиндрична симетрия. За това е естествено техните магнитни полета да бъдат представени във вид на сума от азимутални, $\mathbf{B}_a = \mathbf{1}_\phi B_\phi$, и меридионални, $\mathrm{B}_m = \mathrm{B} - \mathrm{B}_a$, компоненти. Ние извеждаме уравненията за осисиметрични азимутални и меридионални полета, а също така и уравненията за не-осисиметрични полета с нулево средно. Както това е прието, ние изразяваме осисиметричните меридионални полета чрез функцията на тока: $\overline{\mathbf{B}}_m = \mathbf{1}_\phi \times s^{-1} \nabla \psi(s, z)$. Нашите уравнения са изведени за случая на свиваема течност. Въпреки това, в отсъствие на електродвижеща сила те съвпадат по форма с уравненията на свиваемата течност за осисимметрични полета (виж. например Моффатт (1978). Обаче свиваемостта играе съществена роля в тези уравнения. Тя води до поява на нови генерационни членове в уравненията за полоидално и тороидално полета. Базирайки се на нашите уравнения, ние анализираме генерацията на магнитното поле в земното ядро. Този анализ показва, че е възможна ефективна генерация в граничните слоеве на границите между вътрешното и външното ядра и между външното ядро и мантията. Тази генерация възниква благодарение на α и ω ефекти появяващи се когато параметърът $R_m \hat{\delta}$ не е малък. Тук $\hat{\delta}$ е дебелина на слоя, а $\mathbf{R}_m$ е магнитното число на Рейнолдс. В този случай главната част на осисиметричното магнитно поле се поддържа от тази слоева генерация.